On the Diameter of Zero-Divisor Graphs of Idealizations with Respect to Integral Domain

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Abstract

Let $R$ be a ring with unity and let $M$ be an $R$ - module. Let $R(+)M$ be the idealization of the ring $R$ by the $R$ - module $M$. In this paper, we give new results on the diameter of $\Gamma(R(+)M)$ when $R$ is an integral domain.

Introduction

The zero divisor graph of a ring is the (simple) graph whose vertex set is the set of non-zero zero divisors, and an edge is drawn between two distinct vertices if their product is zero. The zero divisor graph of a commutative ring has been studied extensively by several authors, see [1, 2, 3 and 4]. Let $R$ be a commutative ring with unity. We use the notation $A^*$ to refer to the nonzero elements of $A$. For two distinct vertices $a$ and $b$ in a graph $\Gamma(R)$, the distance between $a$ and $b$, denoted by $d(a,b)$, is the length of the shortest path connecting $a$ and $b$, if such a path exists, otherwise, $d(a,b) = \infty$. The diameter of a graph $\Gamma(R)$ is $\text{diam}(\Gamma) = \sup \{d(a,b) : a$ and $b$ are distinct vertices of $\Gamma\}$. We will use the notation $diam(\Gamma(R))$ to denote the diameter of the $\theta$nonzero zero divisors of $R$).

Let $M$ be an $R$ - module. Consider $R(+)M = \{(a, m) : a \in R, m \in M\}$ and let $(a, m)$ and $(b, n)$ be two elements of $R(+)M$. Define $(a, m) + (b, n) = (a + b, m + n)$ and $(a, m)(b, n) = (ab, an + bm)$. Under this definition $R(+)M$ becomes a commutative ring with unity. Call this
ring the idealization ring of $M$ in $R$. For more details, we refer the reader to [5]. The set of all nonzero zero divisors of a ring $R$ is denoted by $Z^*(R)$.

$$Diam(\Gamma(R(+)M)) \text{ when } R \text{ is an Integral Domain}$$

**Lemma 1.** Let $R$ be an integral domain such that $Z_2$ is an $R$-module with $\text{ann}(Z_2) = \{0\}$. Then $R \cong Z_2$.

**Proof.** Now $1_R.1 = 1$ since $Z_2$ is unitary $R$-module. So, $(1_R + 1_R).1 = 1_R.1 + 1_R.1 = 1 + 1 = 0$. Hence $2.1_R = 1_R + 1_R \in \text{ann}(Z_2) = \{0\}$. Hence $ch(R) = 2$.

Moreover, $r.1 = 1$ for all $r \in R^*$ because $\text{ann}(Z_2) = \{0\}$. Assume that $r \in R - \{0_R, 1_R\}$. Then $(r + 1_R).1 = 1 + 1 = 0$, hence $r = -1_R = 1_R$, a contradiction. Then $R \cong Z_2$. ■

**Lemma 2.** Let $R$ be an integral domain such that $Z_3$ is an $R$-module with $\text{ann}(Z_3) = \{0\}$. Then $R \cong Z_3$.

**Proof.** $0, 1_R \in R$ and $1_R \neq 0_R$. Then $2 1_R \in R$. If $2 1_R = 0_R$, then $3 1_R.1 = 0$ because $1_R.1 = (1_R + 1_R + 1_R).1 = 1 + 1 + 1 = 0$. So, $0 = 3 1_R.1 = (2 1_R + 1_R).1 = 2 1_R.1 + 1_R.1 = 0 + 1 = 1$, contradiction. So, $2 1_R \neq 0_R$. If $2 1_R = 1_R$, then $1_R = 0_R$ since $(R, +)$ is abelian group. So, $R$ has at least 3 element $0_R, 1_R$ and $2 1_R$. Assume that there exist, $r \in R - \{0_R, 1_R, 21_R\}$, $r \in R$. If $r.1 = 0$, then $r \in \text{ann}(Z_3)$. Hence $r = 0_R$, contradiction. If $r.1 = 1$, then $(r + 21_R).1 = r.1 + 21_R.1 = r.1 + 1 + 1 = 1 + 1 + 1 = 0$. So, $r + 21_R = 0_R$ since $\text{ann}(Z_3) = \{0\}$. Then $r = -21_R = 1_R$, contradiction. If $r.1 = 2$, then $(r + 1_R).1 = 0$. So, $r + 1_R = 0_R$ i.e $r = -1_R = 21_R$, contradiction. Thus $R = \{0_R, 1_R, 21_R\}$ and $31_R = 0_R$. Then $(R, +) \cong (Z_3, +)$.

\[
(R, \cdot) = \begin{pmatrix}
0_R & 0_R & 0_R & 0_R \\
1_R & 0_R & 1_R & 21_R \\
21_R & 0_R & 21_R & 1_R
\end{pmatrix}
\] Then $R \cong Z_3$ (as a ring). ■
**Theorem 3.** Let $R$ be an integral domain and $M \cong \mathbb{Z}_2$ be an $R$-module.

(i) If $\text{ann}(\mathbb{Z}_2) = \{0\}$, then $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_2)) = 0$.

(ii) If $\text{ann}(\mathbb{Z}_2) \neq \{0\}$, then $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_2)) = 2$.

**Proof.** (i) $M \cong \mathbb{Z}_2$, and $\text{ann}(\mathbb{Z}_2) = \{0\}$. Then, by Lemma (1), $R \cong \mathbb{Z}_2$. Then $Z^*(\mathbb{Z}_2(\oplus)\mathbb{Z}_2) = \{(0,1)\}$, so $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_2)) = 0$.

(ii) $M \cong \mathbb{Z}_2$ and $\text{ann}(\mathbb{Z}_2) \neq \{0\}$. Then there exists at least one element in $R^*$ such that $r.1 = 0$. $Z^*(R(\oplus)\mathbb{Z}_2) = \{(0,1)\} \cup \{(r,0),(r,1),\ldots\}$. Any two elements in $\{(r,0),(r,1),\ldots\}$ are non adjacent, but $(r,0),(0,1) = (0,0)$ and $(0,1),(r,1) = (0,0)$. $(r,0)--(0,1)--(r,1)$ so, $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_2)) = 2$. ■

**Theorem 4.** Let $R$ be an integral domain and $M \cong \mathbb{Z}_3$ be an $R$-module.

(i) If $\text{ann}(\mathbb{Z}_3) = \{0\}$, then $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_3)) = 1$.

(ii) If $\text{ann}(\mathbb{Z}_3) \neq \{0\}$, then $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_3)) = 2$.

**Proof.** (i) $M \cong \mathbb{Z}_3$, $\text{ann}(\mathbb{Z}_3) = \{0\}$. Then, by Lemma (2), $R \cong \mathbb{Z}_3$. So, $Z^*(\mathbb{Z}_3(\oplus)\mathbb{Z}_3) = \{(0,1),(0,2)\}$. Thus $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_3)) = 1$. $\Gamma(R(\oplus)\mathbb{Z}_3)$ is $(0,1)\rightarrow(0,2)$.

(ii) $M \cong \mathbb{Z}_3$ and $\text{ann}(\mathbb{Z}_3) \neq \{0\}$. Then there exists at least one element in $R^*$ such that $r.\mathbb{Z}_3 = 0$. So, $Z^*(R(\oplus)\mathbb{Z}_3) = \{(0,1),(0,2)\} \cup \{(r,0),(r,1),(r,2),\ldots\}$. Any two elements in $\{(r,0),(r,1),(r,2),\ldots\}$ are non adjacent since $r.s \neq 0$ for any $r, s \in R^*$. But $(r,0)--(0,1)--(r,2)$. Hence, $\text{diam}(\Gamma(R(\oplus)\mathbb{Z}_2)) = 2$. ■

**Theorem 5.** Let $R$ be an integral domain and $|M| \geq 4$, be an $R$-module.

(i) If $r.m \neq 0$, for any $r \in R^*$ and $m \in M^*$, then $\text{diam}(\Gamma(R(\oplus)M)) = 1$.

(ii) If there exists at least one element $m \in M^*$ such that $r.m = 0$, for any $r \in R$, then $\text{diam}(\Gamma(R(\oplus)M)) = 2$. 

(iii) If there exists at least two elements in $R^*$ such that $r_1.m = 0$, $r_2.n = 0$, $r_1.n \neq 0$, $r_2.m \neq 0$, $m \neq n$, for $m, n \in M^*$, then $\text{diam}(\Gamma(R(+)M)) = 3$.

**Proof.** (i) If there is no element $r \in R^*$ such that $r.m = 0$, for any $m \in M^*$, then $Z^*(R(+)M) = \{(0, m) : m \in M^*\}$. Then any two elements in $\{(0, m) : m \in M^*\}$ are at distance 1.

(ii) If there exists at least one element $m \in M^*$ such that $r.m = 0$, for any $r \in R$, then $Z^*(R(+)M) = \{(0, m) : m \in M^*\} \cup \{(r, m), \ldots : m \in M\}$. Then any two elements in $\{(r, m), \ldots : m \in M^*\}$ are non adjacent. But $(r, n).(0, m) = (0, 0)$, and $(0, m).(0, n) = (0, 0)$. Then $(r, n) - (0, m) - (0, n), m \neq n$, and $m, n \in M^*$. Then $\text{diam}(\Gamma(R(+)M)) = 2$.

(iii) If there exists two elements $r_1, r_2 \in R^*$ such that $r_1.m = 0$, $r_2.n = 0$, $m \neq n$, and $m, n \in M^*$, then $Z^*(R(+)M) = \{(0, m) : m \in M^*\} \cup \{(r_1, m), (r_2, n), \ldots : m \neq n, m, n \in M^*\}$. Any two elements in $\{(r_1, n), (r_2, m), \ldots : m \neq n, m, n \in M^*\}$ are non adjacent. But $(r_1, 0) - (0, m) - (0, n) - (r_2, 0)$. Then $\text{diam}(\Gamma(R(+)M)) = 3$. ■

**Example 1.** Consider the ring $\mathbb{Z}_5(+)\mathbb{Z}_5$. Then $\text{diam}(\Gamma(R(+)M)) = 1$. Since $\mathbb{Z}_5$ is an integral domain and $\mathbb{Z}_5$ is an $\mathbb{Z}_5$-module, and there is no element in $\mathbb{Z}_5^*$ such that $r.m = 0$, for any $m \in \mathbb{Z}_5^*$.

**Example 2.** Consider the ring $\mathbb{Z}(+)\mathbb{Z}_{18}$. Then $\text{diam}(\Gamma(R(+)M)) = 3$. Since $\mathbb{Z}$ is an integral domain and $\mathbb{Z}_{18}$ is an $\mathbb{Z}$-module, and there exists two elements $(r_1 = 2, r_2 = 9)$ in $\mathbb{Z}^*$ such that $2.9 = 0, 9.2 = 0, 2.2 \neq 0, 9.9 \neq 0, 9, 2 \in \mathbb{Z}_{18}^*$, then $(2, 0) - (0, 9) - (0, 2) - (9, 0)$.

**Example 3.** Consider the ring $\mathbb{Z}(+)\mathbb{Z}_5$. Then $\text{diam}(\Gamma(R(+)M)) = 2$. Since $\mathbb{Z}$ is an integral domain and $\mathbb{Z}_5$ is an $\mathbb{Z}$-module, and there exists $5 \in \mathbb{Z}$ and $5.3 = 0, 3 \in \mathbb{Z}_5$.

And for all $r \in \mathbb{Z}$, $r.3 = 0$. 
References


