Calculus II : For Science and Engineering
Lecture Notes for Calculus 102

Feras Awad Mahmoud

Last Updated: October 14, 2023

Feras Awad Mahmoud
Department of Basic Sciences
Philadelphia University
JORDAN 19392

TEXTBOOK:
This book is strongly recommended for Calculus 102 as well as a reference text for subsequent courses in mathematics. The pdf soft−copy of the three chapters remain available for free download.

TYPESETTING:
The entire document was written in LaTeX, implemented for Windows using the MiKTeX 2.9 distribution. As for the text editor of my choice, I fancy Notepad++ 6.6.8.

MORAL SUPPORT:
My wife with my children Saleem and Sarah have had a very instrumental role in providing moral support. Thank God for their patience, understanding, encouragement, and prayers throughout the long process of writing and editing.

Contents

CHAPTER 1

Techniques of Integration

In Calculus 1, you studied several basic techniques for evaluating simple integrals. In this chapter, you will study other integration techniques, such as integration by parts, that are used to evaluate more complicated integrals. You will also learn how to evaluate improper integrals.

1.1 Basic Integration Rules

In this chapter, you will study several integration techniques that greatly expand the set of integrals to which the basic integration rules can be applied. A major step in solving any integration problem is recognizing which basic integration rule to use. As shown in Example 1.1, slight differences in the integrand can lead to very different solution techniques.

Example 1.1. Evaluate each integral

$$\textbf{a)} \int \frac{1}{x^2+1}\,dx \qquad \textbf{b)} \int \frac{x}{x^2+1}\,dx \qquad \textbf{c)} \int \frac{x^2}{x^2+1}\,dx$$

Solution 1.1. a) $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C.$

b) $$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln\left(1+x^2\right) + C.$$

c) $$\int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1} x + C.$$

Some times you need to use two basic rules to solve a single integral as shown in Example 1.2.

Example 1.2. Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$.

Solution 1.2. Begin by writing the integral as the sum of two integrals. Then apply the Power (Substitution) Rule and the Arcsine Rule, as follows.

$$\begin{aligned}
\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\int_0^1 \frac{2x}{\sqrt{4-x^2}}\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx \\
&= \left[-\sqrt{4-x^2} + 3\sin^{-1}\frac{x}{2}\right]_0^1 \\
&= 2-\sqrt{3}+\frac{\pi}{2}
\end{aligned}$$

Often you need your intelligence in the appropriate substitution to solve the integration. Consider the following three examples.

Example 1.3. Evaluate $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$.

Solution 1.3. Because two different radicals appear in the problem, the substitution $x = u^6$, [6 = Least Common Multiple of 2 and 3] will eliminate both, and you have

$$\begin{aligned}
\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx &= \int \frac{6u^5}{u^3-u^2}\,du = 6\int \frac{u^3}{u-1}\,du \\
&= 6\int \left[u^2+u+1+\frac{1}{u-1}\right] du \\
&= 2u^3+4u^2+6u+6\ln|u-1|+C \\
&= 2\sqrt{x}+4\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C
\end{aligned}$$

Example 1.4. Find $\int \frac{x^2}{16+x^6}\,dx$.

Solution 1.4. Because the denominator can be written in the form $16+x^6 = 4^2+\left(x^3\right)^2$ you can try the substitution $u = x^3$. Then $du = 3x^2dx$ and you have

$$\begin{aligned}\int \frac{x^2}{16+x^6}\,dx &= \frac{1}{3}\int \frac{3x^2}{4^2+\left(x^3\right)^2}\,dx = \frac{1}{3}\int \frac{1}{4^2+u^2}\,du\\ &= \frac{1}{12}\tan^{-1}\frac{u}{4}+C = \frac{1}{12}\tan^{-1}\frac{x^3}{4}+C\end{aligned}$$

Example 1.5. Evaluate $\displaystyle\int \frac{3e^x+5}{2e^x+7}\,dx$.

Solution 1.5. One of the methods to solve this integral is by writing the integral as the sum of two integrals as in Example 1.2. To do this, we find constants α and β such that

$$3e^x+5 = \alpha\left(2e^x+7\right)+\beta\frac{d}{dx}\left(2e^x+7\right) = 2(\alpha+\beta)e^x+7\alpha$$

Comparing the coefficients in both sides of the above equation yields to solve the two equations $2(\alpha+\beta)=3$ and $7\alpha = 5$ which gives us $\alpha = \dfrac{5}{7}$ and $\beta = \dfrac{11}{14}$. So,

$$\begin{aligned}\int \frac{3e^x+5}{2e^x+7}\,dx &= \frac{5}{7}\int \frac{\cancel{2e^x+7}}{\cancel{2e^x+7}}\,dx+\frac{11}{14}\int \frac{2e^x}{2e^x+7}\,dx\\ &= \frac{5}{7}x+\frac{11}{14}\ln\left(2e^x+7\right)+C\end{aligned}$$

Surprisingly, two of the most commonly overlooked integration rules are the Log Rule and the Power (Substitution) Rule. Notice in the next two examples how these two integration rules can be disguised.

Example 1.6. Find $\displaystyle\int \frac{1}{1+e^x}\,dx$.

Solution 1.6. The integral does not appear to fit any of the basic rules. However, multiply both the numerator and the denominator by e^{-x} and then the quotient form suggests the Log Rule as follows.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+e^x}\times\frac{e^{-x}}{e^{-x}}\,dx = -\int \frac{-e^{-x}}{e^{-x}+1} = \ln\left(e^{-x}+1\right)+C$$

Example 1.7. Evaluate $\displaystyle\int (\cot x)\,[\ln(\sin x)]\,dx$.

Solution 1.7. Again, the integral does not appear to fit any of the basic rules. However, considering the two primary choices for u [$u = \cot x$ and $u = \ln(\sin x)$] you can see that the second choice is the appropriate one because

$$du = \frac{\cos x}{\sin x}\,dx = \cot x\,dx$$

So,

$$\int (\cot x)\,[\ln(\sin x)]\;dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\,[\ln(\sin x)]^2 + C$$

Trigonometric identities can often be used to fit integrals to one of the basic integration rules.

Example 1.8. Find $\displaystyle\int \tan^2(2x)\,dx$.

Solution 1.8. Note that $\tan^2 t$ is not in the list of basic integration rules. However, $\sec^2 t$ is in the list. This suggests the trigonometric identity $\tan^2 t = \sec^2 t - 1$.

$$\int \tan^2(2x)\,dx = \int \left[\sec^2(2x) - 1\right]\,dx = \frac{1}{2}\tan(2x) - x + C$$

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $ax^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$. If the leading coefficient is not 1, it helps to factor before completing the square.

Example 1.9. Find $\displaystyle\int \frac{1}{x^2 - 4x + 7}\,dx$.

Solution 1.9. You can write the denominator as the sum of two squares, as follows.

$$x^2 - 4x + 7 = \left(x^2 - 4x + 4\right) - 4 + 7 = (x-2)^2 + 3$$

Now, in this completed square form, we have

$$\int \frac{1}{x^2 - 4x + 7}\,dx = \int \frac{1}{(x-2)^2 + 3}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x-2}{\sqrt{3}} + C$$

Exercise 1.1. Evaluate each of the following integrals.

1. $\displaystyle\int \frac{1}{(5x-3)^4}\,dx.$

2. $\displaystyle\int \frac{1}{\sqrt{x}\left(1-2\sqrt{x}\right)}\,dx.$ **Hint:** Let $u = 1-2\sqrt{x}$

3. $\displaystyle\int \frac{\ln\left(x^2\right)}{x}\,dx.$ **Hint:** Let $u = \ln\left(x^2\right) = 2\ln x$

4. $\displaystyle\int \frac{6}{\sqrt{10x-x^2}}\,dx.$ **Hint:** Complete the square of $10x - x^2$

5. $\displaystyle\int \sqrt{e^x-1}\,dx.$ **Hint:** Let $u^2 = e^x - 1$

6. $\displaystyle\int_0^6 \frac{2x+5}{\sqrt{2x+4}}\,dx$

7. $\displaystyle\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}\,dx$

8. $\displaystyle\int \frac{e^{2x}-1}{e^{2x}+1}.$

9. $\displaystyle\int \frac{1}{x^{10}+x}\,dx.$ **Hint:** Multiply by $\dfrac{x^{-10}}{x^{-10}}$

10. $\displaystyle\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,dx.$ **Hint:** Multiply by $\dfrac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$

11. $\displaystyle\int \left[\frac{x}{(x-1)^2+1}\right]^2 dx.$ **Hint:** Note that $x^2 = [(x-1)+1]^2$

12. $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}\,dx.$

    Hint: As x goes from 2 to 4, $9-x$ and $x+3$ go from 7 to 5, and from 5 to 7, respectively. This symmetry suggests the substitution $x = 6-y$ reversing the interval $[2,4]$.

1.2 Integration by Parts

In this section you will study an important integration technique called **integration by parts**. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of algebraic and transcendental functions. For instance, integration by parts works well with integrals such as

$$\int x^n \ln x\, dx, \quad \int x^n \sin^{-1} x\, dx, \quad \int x^n e^{ax}\, dx, \text{ and } \int e^{ax} \sin(bx)\, dx$$

Integration by parts is based on the formula for the derivative of a product of two functions $f(x)$ and $g(x)$.

Theorem 1.2.1. *If u and v are functions of x and have continuous derivatives, then*

$$\int u\, dv = u\, v - \int v\, du.$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and v it may be easier to evaluate the second integral than the original one. However, some authors suggest a way for selecting the first and second function. If we denote **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, and **E**xponential functions by their first alphabet respectively, then the first function u is selected according to the letters of the group **LIATE**.

Example 1.10. Evaluate $\int xe^x\, dx$.

Solution 1.10. The LIATE suggests $u = x$ as the first option and $dv = e^x dx$. So,

$$u = x \to du = dx \quad \text{and} \quad dv = e^x dx \to v = e^x$$

Now, integration by parts produces

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$$

Example 1.11. Find $\int 2x^2 \ln \sqrt{x}\, dx$.

Solution 1.11. First notice that

$$\int 2x^2 \ln \sqrt{x}\, dx = \int 2x^2 \ln\left(x^{\frac{1}{2}}\right) dx = \int x^2 \ln x\, dx$$

In this case, we let

$$u = \ln x \to du = \frac{1}{x} dx \quad \text{and} \quad dv = x^2 dx \to v = \frac{x^3}{3}$$

Integration by parts produces

$$\begin{aligned}
\int 2x^2 \ln \sqrt{x}\, dx &= \int x^2 \ln x\, dx \\
&= \frac{1}{3} x^3 \ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right) dx \\
&= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2\, dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C
\end{aligned}$$

Example 1.12. Evaluate $\int_0^1 \sin^{-1} x\, dx$.

Solution 1.12. Let $u = \sin^{-1} x \to du = \dfrac{1}{\sqrt{1-x^2}} dx$ and $dv = dx \to v = x$. Integration by parts now produces

$$\begin{aligned}
\int \sin^{-1} x\, dx &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx \\
&= x \sin^{-1} x + \frac{1}{2} \int \frac{2x}{\sqrt{1-x^2}}\, dx \\
&= x \sin^{-1} x + \sqrt{1-x^2} + C
\end{aligned}$$

Using this anti-derivative, you can evaluate the definite integral as follows.

$$\int_0^1 \sin^{-1} x\, dx = \left[x \sin^{-1} x + \sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - 1$$

Some integrals require starting by substitution method then integrate by parts, may repeatedly.

Example 1.13. Find $\int \frac{1}{3} \sin \sqrt[3]{x}\, dx$.

Solution 1.13. First we use the substitution $x = y^3 \to dx = 3y^2 dy$ to solve this integral, and we obtain

$$\int \frac{1}{3} \sin \sqrt[3]{x}\, dx = \int y^2 \sin y\, dy$$

Let $u = y^2 \to du = 2y dy$ and $dv = \sin y dy \to v = -\cos y$. Integration by parts now produces

$$\int y^2 \sin y\, dy = -y^2 \cos y + \int 2y \cos y\, dy$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2y \to du = 2dy$ and $dv = \cos y dy \to v = \sin y$. Now, integration by parts produces

$$\int 2y \cos y\, dy = 2y \sin y - \int 2 \sin y dy = 2y \sin y + 2 \cos y + C$$

Combining these two results, you can write

$$\begin{aligned}
\int \frac{1}{3} \sin \sqrt[3]{x}\, dx &= \int y^2 \sin y\, dy \\
&= -y^2 \cos y + 2y \sin y + 2 \cos y + C \\
&= -\sqrt[3]{x^2} \cos \sqrt[3]{x} + 2\sqrt[3]{x} \sin \sqrt[3]{x} + 2 \cos \sqrt[3]{x} + C
\end{aligned}$$

The following example will require a technique that deserves special attention.

Example 1.14. Evaluate $\int e^x \cos x\, dx$.

Solution 1.14. Let $u = \cos x \to du = -\sin x dx$ and $dv = e^x dx \to v = e^x$. Thus,

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$$

Since the integral $\int e^x \sin x\, dx$ is similar in form to the original integral $\int e^x \cos x\, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let $u = \sin x \to du = \cos x dx$ and $dv = e^x dx \to v = e^x$. Thus,

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$$

Combining these two results, you can write

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$

which is an equation we can solve for the unknown integral. We obtain

$$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x\,dx = \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + C$$

Example 1.15. Find $\int \sec^3 x\,dx$.

Solution 1.15. The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$ so you should let $u = \sec x \to du = \sec x \tan x dx$ and $dv = \sec^2 x dx \to v = \tan x$. Integration by parts produces

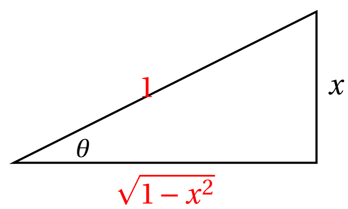
$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\
2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C
\end{aligned}$$

In each of the following problems, the integration by parts is a bit more challenging.

Example 1.16. Evaluate $\int \left(\sin^{-1} x\right)^2\,dx$.

Solution 1.16. Let $\theta = \sin^{-1} x$. So, $x = \sin\theta$ and $dx = \cos\theta\, d\theta$. Thus,

$$
\begin{aligned}
\int \left(\sin^{-1} x\right)^2 dx &= \int \theta^2 \cos\theta\, d\theta \quad (\text{let } u = \theta^2 \text{ and } dv = \cos\theta\, d\theta) \\
&= \theta^2 \sin\theta - \int 2\theta \sin\theta\, d\theta \quad (\text{let } u = 2\theta \text{ and } dv = \sin\theta\, d\theta) \\
&= \theta^2 \sin\theta + 2\theta\cos\theta - 2\int \cos\theta\, d\theta \\
&= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C \\
&= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x - 2x + C
\end{aligned}
$$



Example 1.17. Evaluate $\displaystyle\int \frac{x^2 e^x}{(x+2)^2}\, dx$.

Solution 1.17. let $u = x^2 e^x$ and $dv = \dfrac{1}{(x+2)^2}\, dx$

$$
\begin{aligned}
\int \frac{x^2 e^x}{(x+2)^2}\, dx &= -\frac{x^2 e^x}{x+2} + \int \frac{\cancel{(x+2)}\, x e^x}{\cancel{x+2}}\, dx \\
&= -\frac{x^2 e^x}{x+2} + \int x e^x\, dx \\
&= -\frac{x^2 e^x}{x+2} + x e^x - e^x + C
\end{aligned}
$$

Exercise 1.2. Evaluate each of the following integrals.

1. $\displaystyle\int \left(x^2 - x + 1\right) e^x\, dx$. **Hint:** by parts, let $u = x^2 - x + 1$

2. $\displaystyle\int x\sqrt{x-5}\, dx$. **Hint:** by substitution, let $y = x - 5$

3. $\displaystyle\int_0^{\pi/8} x \sec^2 x\, dx$. **Hint:** by parts, let $u = x$

4. $\int \cos(\ln x)\,dx$. **Hint:** Start by substituting $y = \ln x$

5. $\int xe^x \sin x\,dx$. **Hint:** by parts, let $u = x$

6. $\int \ln\left(x + \sqrt{x^2+1}\right)dx$. **Hint:** by parts, let $u = \ln\left(x + \sqrt{x^2+1}\right)$

7. $\int \frac{x}{1+\sin x}\,dx$. **Hint:** Multiply by $\frac{1-\sin x}{1-\sin x}$

8. $\int \frac{\ln x - 1}{(\ln x)^2}\,dx$.

9. $\int x(1+\ln x)^2\,dx$. **Hint:** by parts, let $u = (1+\ln x)^2$

10. $\int (\ln 2x)(\ln x)\,dx$. **Hint:** $\ln(ab) = \ln a + \ln b$

11. $\int \ln\left(\frac{x+1}{x-1}\right)dx$. **Hint:** $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

12. $\int \sqrt{x}\tan^{-1}\sqrt{x}\,dx$.

1.3 Trigonometric Integrals

In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x\,dx \qquad \text{and} \qquad \int \sec^m \tan^n\,dx$$

where either m or n is a positive integer. To find anti-derivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule. To break up $\int \sin^m x \cos^n x\,dx$ into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Algorithm 1.1. Guidelines for Evaluating Integrals Involving Powers of Sine and Cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\begin{aligned}\int \sin^{\overbrace{2k+1}^{\textbf{Odd}}} x \cos^n x\,dx &= \int \overbrace{\left(\sin^2 x\right)^k}^{\textbf{Convert to cos}} \cos^n x \overbrace{\sin x\,dx}^{\textbf{Save for } du} \\ &= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x\,dx\end{aligned}$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\begin{aligned}\int \sin^m x \cos^{\overbrace{2k+1}^{\textbf{Odd}}} x\,dx &= \int \sin^m x \overbrace{\left(\cos^2 x\right)^k}^{\textbf{Convert to sin}} \overbrace{\cos x\,dx}^{\textbf{Save for } du} \\ &= \int \sin^m x \left(1-\sin^2 x\right)^k \cos x\,dx\end{aligned}$$

3. If the powers of both the sine and cosine are even and non-negative, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

   to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

Example 1.18. Evaluate $\int \sin^3 x \cos^4 x\,dx$.

Solution 1.18. Because you expect to use the Power Rule with $u = \cos x$, save one sine factor to form du and convert the remaining sine factors to cosines.

$$\begin{aligned}
\int \sin^3 x \cos^4 x\,dx &= \int \sin^2 x \cos^4 x \sin x\,dx \\
&= \int \left(1-\cos^2 x\right)\cos^4 x \sin x\,dx \\
&= \int \left(\cos^4 x - \cos^6 x\right)\sin x\,dx \quad \text{Let } u=\cos x \\
&= \int \left(u^6 - u^4\right)\,du \\
&= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\
&= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

In the next example the power of the cosine is 3, but the power of the sine is $-\frac{1}{2}$.

Example 1.19. Find $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$.

Solution 1.19. Because you expect to use the Power Rule with $u=\sin x$, save one cosine factor to form du and convert the remaining cosine factors to sines.

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx &= \int_{\pi/6}^{\pi/2} \frac{\cos^2 x \cos x}{\sqrt{\sin x}}\,dx \\
&= \int_{\pi/6}^{\pi/2} \frac{\left(1-\sin^2 x\right)\cos x}{\sqrt{\sin x}}\,dx \quad \text{Let } u=\sin x \\
&= \int_{1/2}^{1} u^{-\frac{1}{2}}\left(1-u^2\right)\,du \\
&= \int_{1/2}^{1} \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}}\right)\,du = \left[2u^{\frac{1}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_{1/2}^{1} = \frac{32-19\sqrt{2}}{20}
\end{aligned}$$

Example 1.20. Evaluate $\displaystyle\int \cos^4 x\,dx$.

Solution 1.20. Because m and n are both even and non-negative ($m = 0$) you can replace $\cos^4 x$ by $\left[\frac{1+\cos(2x)}{2}\right]^2$.

$$\begin{aligned}
\int \cos^4 x\,dx &= \int \left[\frac{1+\cos(2x)}{2}\right]^2 dx \\
&= \int \left[\frac{1}{4} + \frac{\cos(2x)}{2} + \frac{\cos^2(2x)}{4}\right] dx \\
&= \int \left[\frac{1}{4} + \frac{\cos(2x)}{2} + \frac{1+\cos(4x)}{8}\right] dx \\
&= \frac{3}{8}\int dx + \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{8}\int \cos(4x)\,dx \\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Theorem 1.3.1. ***WALLIS'S FORMULAS***

1. *If n is odd $(n \geq 3)$, then*

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

2. *If n is even $(n \geq 2)$, then*

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$.

Example 1.21. Evaluate $\displaystyle\int_0^{\pi/2} \left(8\cos^4 x - 3\sin^5 x\right) dx$.

Solution 1.21. By using Wallis's Formulas, we have

$$\begin{aligned}
\int_0^{\pi/2} \left(8\cos^4 x - 3\sin^5 x\right) dx &= 8\int_0^{\pi/2} \cos^4 x\,dx - 3\int_0^{\pi/2} \sin^5 x\,dx \\
&= 8\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{\pi}{2}\right) - 3\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) \\
&= \frac{15\pi - 16}{10}
\end{aligned}$$

The following guidelines can help you evaluate integrals of the form $\int \sec^m x \tan^n x\,dx$.

Algorithm 1.2. Guidelines for Evaluating Integrals Involving Powers of Secant and Tangant

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\begin{aligned}\int \sec^{\overbrace{2k}^{\textbf{Even}}} x \tan^n x\,dx &= \int \overbrace{\left(\sec^2 x\right)^{k-1}}^{\textbf{Convert to tan}} \tan^n x \overbrace{\sec^2 x\,dx}^{\textbf{Save for } du} \\ &= \int \left(1+\tan^2 x\right)^{k-1} \tan^n x \sec^2 x\,dx\end{aligned}$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\begin{aligned}\int \sec^m x \tan^{\overbrace{2k+1}^{\textbf{Odd}}} x\,dx &= \int \sec^{m-1} x \overbrace{\left(\tan^2 x\right)^k}^{\textbf{Convert to sec}} \overbrace{\sec x \tan x\,dx}^{\textbf{Save for } du} \\ &= \int \sec^{m-1} x \left(\sec^2 x - 1\right)^k \sec x \tan x\,dx\end{aligned}$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\begin{aligned}\int \tan^n x\,dx &= \int \tan^{n-2} x \overbrace{\left(\tan^2 x\right)}^{\textbf{Convert to sec}} dx \\ &= \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx\end{aligned}$$

4. If the integral is of the form $\int \sec^m x\,dx$ where m is odd and positive, use integration by parts, as illustrated in Example 1.15 in the preceding section.

5. If none of the above applies, try converting to sines and cosines.

Example 1.22. Evaluate $\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$.

Solution 1.22. Because you expect to use the Power Rule with $u = \sec x$, save a factor of $\sec x \tan x$ to form du and convert the remaining tangent factors to secants.

$$\begin{aligned}
\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx &= \int (\sec x)^{-\frac{1}{2}} \tan^3 x\, dx \\
&= \int (\sec x)^{-\frac{3}{2}} \tan^2 x \sec x \tan x\, dx \\
&= \int (\sec x)^{-\frac{3}{2}} \left(\sec^2 x - 1\right) \sec x \tan x\, dx \quad \text{Let } u = \sec x \\
&= \int u^{-\frac{3}{2}} \left(u^2 - 1\right)\, du = \int \left(u^{\frac{1}{2}} - u^{-\frac{3}{2}}\right) du \\
&= \frac{2}{3} u^{\frac{3}{2}} + 2u^{-\frac{1}{2}} + C \\
&= \frac{2}{3} \sec^{\frac{3}{2}} x + 2\sec^{-\frac{1}{2}} x + C
\end{aligned}$$

Example 1.23. Find $\int \sec^4(3x) \tan^3(3x)\, dx$.

Solution 1.23. Let $u = \tan(3x)$ then $du = 3\sec^2(3x) dx$ and you can write

$$\begin{aligned}
\int \sec^4(3x) \tan^3(3x)\, dx &= \int \sec^2(3x) \tan^3(3x) \sec^2(3x)\, dx \\
&= \int \left(1 + \tan^2(3x)\right) \tan^3(3x) \sec^2(3x)\, dx \\
&= \frac{1}{3} \int \left(1 + u^2\right) u^3\, du = \frac{1}{3} \int \left(u^3 + u^5\right) du \\
&= \frac{1}{12} u^4 + \frac{1}{18} u^6 + C \\
&= \frac{1}{12} \tan^4(3x) + \frac{1}{18} \tan^6(3x) + C
\end{aligned}$$

Example 1.24. Evaluate $\int_0^{\pi/4} \tan^4 x\, dx$.

Solution 1.24. Because there are no secant factors, you can begin by converting a tangent-squared factor to a secant-squared factor.

$$\begin{aligned}\int \tan^4 x\,dx &= \int \tan^2 x \tan^2 x\,dx = \int \tan^2 x\left(\sec^2 x - 1\right)\,dx\\ &= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx\\ &= \int \tan^2 x \sec^2 x\,dx - \int \left(\sec^2 x - 1\right)\,dx\\ &= \frac{1}{3}\tan^3 x - \tan x + x + C\end{aligned}$$

You can evaluate the definite integral as follows.

$$\int_0^{\pi/4} \tan^4 x\,dx = \left[\frac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \frac{\pi}{4} - \frac{2}{3}$$

For integrals involving powers of cotangents and cosecants, you can follow a strategy similar to that used for powers of tangents and secants. Also, when integrating trigonometric functions, remember that it sometimes helps to convert the entire integrand to powers of sines and cosines.

Example 1.25. Find $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$.

Solution 1.25. Because the guidelines do not apply, try converting the integrand to sines and cosines. In this case, you are able to integrate the resulting powers of sine and cosine as follows.

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x}\,dx &= \int \left(\frac{1}{\cos x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right)\,dx\\ &= \int \frac{\cos x}{\sin^2 x}\,dx \quad \text{Let } u = \sin x \to du = \cos x\,dx\\ &= \int \frac{1}{u^2}\,du = -\frac{1}{u} + C\\ &= -\frac{1}{\sin x} + C = -\csc x + C\end{aligned}$$

Integrals involving the products of sines and cosines of two **different angles** occur in many applications. In such instances you can use the following product-to-sum identities.

$$
\begin{aligned}
\sin(mx)\sin(nx) &= \frac{1}{2}\left\{\cos\left[(m-n)x\right] - \cos\left[(m+n)x\right]\right\} \\
\sin(mx)\cos(nx) &= \frac{1}{2}\left\{\sin\left[(m-n)x\right] + \sin\left[(m+n)x\right]\right\} \\
\cos(mx)\cos(nx) &= \frac{1}{2}\left\{\cos\left[(m-n)x\right] + \cos\left[(m+n)x\right]\right\}
\end{aligned}
$$

Example 1.26. Find $\int \sin(5x)\cos(4x)\,dx$.

Solution 1.26. Considering the second product-to-sum identity above, you can write

$$
\begin{aligned}
\int \sin(5x)\cos(4x)\,dx &= \frac{1}{2}\int (\sin x + \sin(9x))\,dx \\
&= -\frac{1}{2}\cos x - \frac{1}{18}\cos(9x) + C
\end{aligned}
$$

Exercise 1.3. Evaluate the following integrals.

1. $\int \sin^5 x\,dx$
2. $\int \sin^5 x \cos x\,dx$
3. $\int \sin x \tan^2 x\,dx$
4. $\int x \sin^2 x\,dx$
5. $\int \left(\tan^4 x - \sec^4 x\right)\,dx$
6. $\int \frac{\cos(2x)}{\cos x}\,dx$
7. $\int_0^{\pi/2} \sin^{12} x\,dx$
8. $\int \sin(-4x)\sin(3x)\,dx$

1.4 Trigonometric Substitutions

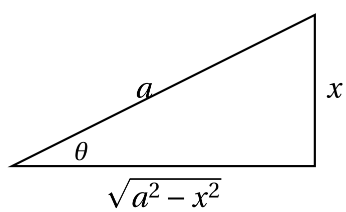
Now that you can evaluate integrals involving powers of trigonometric functions, you can use trigonometric substitution to evaluate integrals involving the radicals $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$. The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

$$\cos^2\theta = 1-\sin^2\theta, \quad \sec^2\theta = 1+\tan^2\theta, \quad \tan^2\theta = \sec^2\theta - 1$$

Note 1.1. TRIGONOMETRIC SUBSTITUTION

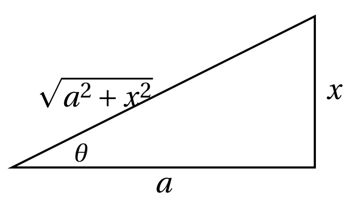
1. For integrals involving $\sqrt{a^2-x^2}$, let $x = a\sin\theta$. Then

$$\sqrt{a^2-x^2} = a\cos\theta \text{ where } -\pi/2 \le \theta \le \pi/2$$



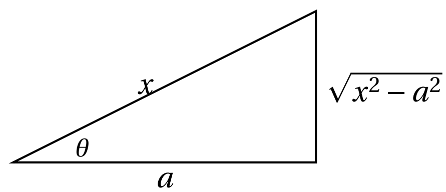
2. For integrals involving $\sqrt{a^2+x^2}$, let $x = a\tan\theta$. Then

$$\sqrt{a^2+x^2} = a\sec\theta \text{ where } -\pi/2 < \theta < \pi/2$$



3. For integrals involving $\sqrt{x^2-a^2}$, let $x = a\sec\theta$. Then

$$\sqrt{x^2-a^2} = \begin{cases} a\tan\theta & \text{if } x > a \quad \text{where } 0 \le \theta < \pi/2 \\ -a\tan\theta & \text{if } x < -a \quad \text{where } \pi/2 < \theta \le \pi \end{cases}$$



The restrictions on θ ensure that the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.

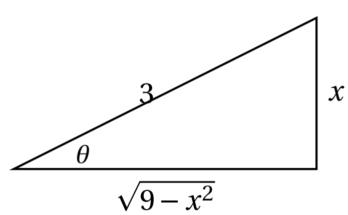
Example 1.27. Find $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$.

Solution 1.27. First, note that none of the basic integration rules applies. To use trigonometric substitution, you should observe that $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-x^2}$. So, you can use the substitution $x = a\sin\theta = 3\sin\theta$. Using differentiation and the triangle shown below, you obtain

$$dx = 3\cos\theta\, d\theta, \quad \sqrt{9-x^2} = 3\cos\theta, \quad x^2 = 9\sin^2\theta$$

So, trigonometric substitution yields

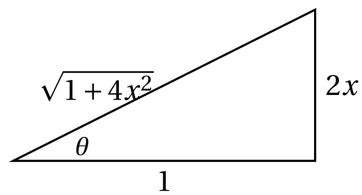
$$\begin{aligned}\int \frac{1}{x^2\sqrt{9-x^2}}\,dx &= \int \frac{3\cos\theta}{\left(9\sin^2\theta\right)(3\cos\theta)}\,d\theta \\ &= \frac{1}{9}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \csc^2\theta\, d\theta = -\frac{1}{9}\cot\theta + C \\ &= -\frac{\sqrt{9-x^2}}{9x} + C\end{aligned}$$



Example 1.28. Find $\int \frac{1}{\sqrt{4x^2+1}}\,dx$.

Solution 1.28. Let $2x = \tan\theta$ then $dx = \dfrac{1}{2}\sec^2\theta\, dx$ and $\sqrt{4x^2+1} = \sec\theta$. Trigonometric substitution produces

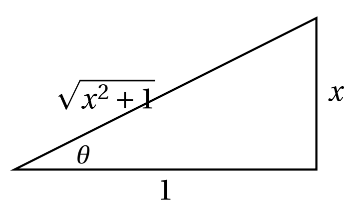
$$\begin{aligned}\int \frac{1}{\sqrt{4x^2+1}}\,dx &= \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta \\ &= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right| + C\end{aligned}$$



Example 1.29. Evaluate $\displaystyle\int \frac{1}{\left(x^2+1\right)^{3/2}}\,dx$.

Solution 1.29. Begin by writing $\left(x^2+1\right)^{3/2}$ as $\left(\sqrt{x^2+1}\right)^3$. Then, let $x = \tan\theta$. Using $dx = \sec^2\theta d\theta$ and $\sqrt{x^2+1} = \sec\theta$ you can apply trigonometric substitution, as follows.

$$\begin{aligned}\int \frac{1}{\left(x^2+1\right)^{3/2}}\,dx &= \int \frac{1}{\left(\sqrt{x^2+1}\right)^3}\,dx = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta \\ &= \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C \\ &= \frac{x}{\sqrt{x^2+1}} + C\end{aligned}$$



For definite integrals, it is often convenient to determine the integration limits for θ that avoid converting back to x.

Example 1.30. Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Solution 1.30. Because $\sqrt{x^2-3}$ has the form $\sqrt{x^2-a^2}$, you can consider $x = \sqrt{3}\sec\theta$. Then $dx = \sqrt{3}\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-3} = \sqrt{3}\tan\theta$. To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3}\sec\theta$ as follows.

$$\text{when } x = \sqrt{3} \to \sec\theta = 1 \to \theta = 0$$
$$\text{when } x = 2 \to \sec\theta = \frac{2}{\sqrt{3}} \to \theta = \frac{\pi}{6}$$

So, you have

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_0^{\pi/6} \frac{\left(\sqrt{3}\tan\theta\right)\left(\sqrt{3}\sec\theta\tan\theta\right)}{\sqrt{3}\sec\theta}\,d\theta \\
&= \int_0^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta \\
&= \sqrt{3}\int_0^{\pi/6} \left(\sec^2\theta - 1\right)\,d\theta \\
&= \sqrt{3}\,[\tan\theta - \theta]_0^{\pi/6} = 1 - \frac{\sqrt{3}\pi}{6}
\end{aligned}$$

Exercise 1.4. Evaluate the following integrals.

1. $\displaystyle\int x\sqrt{1+x^2}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{49-x^2}}\,dx$
3. $\displaystyle\int (x+1)\sqrt{x^2+2x+2}\,dx$
4. $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$
5. $\displaystyle\int \frac{1}{4+4x^2+x^4}\,dx$
6. $\displaystyle\int \sqrt{\frac{1-x}{x}}\,dx$

7. $\int \sqrt{1-e^{2x}}\,dx$

8. $\int \cos x\sqrt{4\sin^2 x+9}\,dx$

9. $\int \sqrt{\dfrac{x-1}{x+1}}\,dx$. **Hint:** Multiply by $\sqrt{\dfrac{x-1}{x-1}}$

1.5 Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the **method of partial fractions**. Its use depends on the ability to factor the denominator, and to find the partial fractions.

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors. For instance, the polynomial x^5+x^4-x-1 can be written as

$$x^5+x^4-x-1=(x-1)(x+1)^2\left(x^2+1\right)$$

where $(x-1)$ is a linear factor, $(x+1)^2$ is a repeated linear factor, and $\left(x^2+1\right)$ is an irreducible quadratic factor. Using this factorization, you can write the partial fraction decomposition of the rational expression as follows

$$\frac{P(x)}{x^5+x^4-x-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{Dx+E}{x^2+1}$$

where $P(x)$ is a polynomial of degree less than 5, and A,B,C,D,E are constants.

Note 1.2. Decomposition of $\mathbf{\dfrac{N(x)}{D(x)}}$ Into Partial Fractions

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)}=(\text{a polynomial})+\frac{N^*(x)}{D(x)}$$

where the degree of $N^*(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $\dfrac{N^*(x)}{D(x)}$.

2. **Factor denominator:** Completely factor the denominator into factors of the form $(\alpha x+\beta)^m$ and $\left(ax^2+bx+c\right)^n$ where ax^2+bx+c is irreducible.

3. **Linear factors:** For each factor of the form $(\alpha x+\beta)^m$ the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(\alpha x+\beta)}+\frac{A_2}{(\alpha x+\beta)^2}\cdots+\frac{A_m}{(\alpha x+\beta)^m}$$

4. **Quadratic factors:** For each factor of the form $\left(ax^2+bx+c\right)^n$ the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{\left(ax^2+bx+c\right)}+\frac{B_2x+C_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{B_nx+C_n}{\left(ax^2+bx+c\right)^n}$$

Example 1.31. Find $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$.

Solution 1.31. Because $x^2-5x+6=(x-3)(x-2)$ you should include one partial fraction for each factor and write

$$\frac{1}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}$$

where A and B are to be determined. Multiplying this equation by the least common denominator $(x-3)(x-2)$ yields the **basic equation**

$$1=A(x-2)+B(x-3)$$

Because this equation is to be true for all x, you can substitute any convenient values for x to obtain equations in A and B. The most convenient values are the ones that make particular factors equal to 0. To solve for A, let $x=3$ to obtain $A=1$. To solve for B, let $x=2$ to obtain $B=-1$. So,

$$\begin{aligned}\int \frac{1}{x^2-5x+6}\,dx &= \int\left[\frac{1}{x-3}-\frac{1}{x-2}\right]dx\\ &= \ln|x-3|-\ln|x-2|+C=\ln\left|\frac{x-3}{x-2}\right|+C\end{aligned}$$

Example 1.32. Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$.

Solution 1.32. Because $x^3+2x^2+x = x(x+1)^2$ you should include one fraction for each power of x and $x+1$ and write

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Multiplying by the least common denominator $x(x+1)^2$ yields the basic equation

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

To solve for A let $x = 0$. This eliminates the B and C terms and yields $A = 6$. To solve for C let $x = -1$. This eliminates the A and B terms and yields $C = 9$. The most convenient choices for x have been used, so to find the value of B, you can use any other value of x along with the calculated values of A and C. Using $x = 1$, $A = 6$, and $C = 9$ produces $B = -1$. So, it follows that

$$\begin{aligned}\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx &= \int \left[\frac{6}{x} - \frac{1}{(x+1)} + \frac{9}{(x+1)^2}\right] dx \\ &= 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C \\ &= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C\end{aligned}$$

When using the method of partial fractions with **linear** factors, a convenient choice of x immediately yields a value for one of the coefficients. With **quadratic** factors, a system of linear equations usually has to be solved, regardless of the choice of x.

Example 1.33. Find $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$.

Solution 1.33. Because $\left(x^2-x\right)\left(x^2+4\right) = x(x-1)\left(x^2+4\right)$ you should include one partial fraction for each factor and write

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

Multiplying by the least common denominator $x(x-1)\left(x^2+4\right)$ yields the basic equation

$$2x^3-4x-8 = A(x-1)\left(x^2+4\right) + Bx\left(x^2+4\right) + (Cx+D)x(x-1)$$

To solve for A, let $x = 0$ and obtain $A = 2$. To solve for B, let $x = 1$ and obtain $B = -2$. At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations. If $x = -1$, then, using $A = 2$ and $B = -2$ you can obtain $-C + D = 2$. If $x = 2$, you have $2C + D = 8$. Solving these two linear equations yields $C = 2$ and consequently $D = 4$. It follows that

$$\begin{aligned}\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx &= \int \left[\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right] dx \\ &= 2\ln|x| - 2\ln|x-1| + \ln\left(x^2+4\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

An improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function.

Example 1.34. Find $\displaystyle\int \frac{x^3+x^2-1}{x^2+1}\,dx$.

Solution 1.34. The integrand is an improper rational function since the numerator has degree 3 and the denominator has degree 2. Thus, we first perform the long division.

$$\begin{array}{r@{}l} & \qquad\qquad x+1 \\ x^2+1 \big) & \overline{\;x^3+x^2\qquad -1} \\ & \underline{-x^3\qquad -x} \\ & \qquad x^2-x-1 \\ & \qquad \underline{-x^2\qquad -1} \\ & \qquad\qquad -x-2 \end{array}$$

It follows that the integrand can be expressed as

$$\frac{x^3+x^2-1}{x^2+1} = x+1-\frac{x+2}{x^2+1}$$

and hence

$$\begin{aligned}\int \frac{x^3+x^2-1}{x^2+1}\,dx &= \int \left[x+1-\frac{x+2}{x^2+1}\right] dx \\ &= \int x\,dx + \int 1\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 + x - \frac{1}{2}\ln\left(x^2+1\right) - 2\tan^{-1}x + C\end{aligned}$$

Some times it is not necessary to use the partial fractions technique on all rational functions like in the previous example. Also, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following example.

Example 1.35. Evaluate $\displaystyle\int \frac{x^2-x-2}{x^3-2x-4}\,dx$.

Solution 1.35.

$$\begin{aligned}\int \frac{x^2-x-2}{x^3-2x-4}\,dx &= \int \frac{(x+1)\cancel{(x-2)}}{\cancel{(x-2)}\left(x^2+2x+2\right)}\,dx\\ &= \frac{1}{2}\int \frac{2(x+1)}{\left(x^2+2x+2\right)}\,dx = \frac{1}{2}\ln\left(x^2+2x+2\right)+C\end{aligned}$$

Finally, partial fractions can be used with some quotients involving transcendental functions.

Example 1.36. Find $\displaystyle\int \frac{\cos x}{\sin x(\sin x-1)}\,dx$.

Solution 1.36. Let $u=\sin x \to du=\cos x dx$. So,

$$\begin{aligned}\int \frac{\cos x}{\sin x(\sin x-1)}\,dx &= \int \frac{1}{u(u-1)}\,du\\ &= \int \left[\frac{1}{u-1}-\frac{1}{u}\right]du \quad \leftarrow \textbf{By Partial Fractions}\\ &= \ln|u-1|-\ln|u|+C=\ln\left|\frac{u-1}{u}\right|+C\\ &= \ln\left|\frac{\sin x-1}{\sin x}\right|+C=\ln|1-\csc x|+C\end{aligned}$$

The previous example involves a rational expression of $\sin x$ and $\cos x$. If you are unable to find an appropriate method to solve an integral of this forms, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

Note 1.3. Substitution for Rational Functions of Sine and Cosine For integrals involving rational functions of sine and cosine, the substitution

$$u=\frac{\sin x}{1+\cos x}=\tan\left(\frac{x}{2}\right)$$

yields

$$\cos x = \frac{1-u^2}{1+u^2}, \quad \sin x = \frac{2u}{1+u^2}, \quad \text{and} \quad dx = \frac{2}{1+u^2}du$$

Example 1.37. Find $\displaystyle\int \frac{1}{1-\sin x+\cos x}\,dx$.

Solution 1.37. The integrand is a rational function of $\sin x$ and $\cos x$ that does not match any appropriate we have learned before, so we make the substitution $u = \tan(x/2)$. Thus, from Note 1.3 we obtain

$$\begin{aligned}
\int \frac{1}{1-\sin x+\cos x}\,dx &= \int \frac{1}{1-\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du \\
&= \int \frac{2}{\left(1+u^2\right)-2u+\left(1+u^2\right)}\,du \\
&= \int \frac{1}{1-u}\,du = -\ln|1-u|+C = -\ln|1-\tan(x/2)|+C
\end{aligned}$$

Exercise 1.5. Evaluate the following integrals.

1. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$
2. $\displaystyle\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$
3. $\displaystyle\int \frac{2x-1}{(x+1)^3}\,dx$. **Hint:** $2x-1 = 2(x+1)-3$.
4. $\displaystyle\int \frac{\sin x}{\sin x+\tan x}\,dx$.
5. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^3}\,dx$.
6. $\displaystyle\int \frac{1}{\sqrt{x}\left(1+\sqrt[3]{x}\right)^2}\,dx$
7. $\displaystyle\int \frac{\cos x}{\sin^2 x+3\sin x+2}\,dx$
8. $\displaystyle\int \frac{e^{2x}}{(e^x+1)^3}\,dx$

9. $\displaystyle\int \frac{x^5}{(x-1)^{10}(x+1)^{10}}\,dx$

10. $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$

1.6 Improper Integrals

The definition of a definite integral $\displaystyle\int_a^b f(x)\,dx$ requires that the interval $[a,b]$ be finite. Furthermore, the Fundamental Theorem of Calculus, by which you have been evaluating definite integrals, requires that f be continuous on $[a,b]$.

In this section you will study a procedure for evaluating integrals that do not satisfy these requirements, usually because either one or both of the limits of integration are infinite, or f has a finite number of infinite discontinuities in the interval $[a,b]$. Integrals that possess either property are **improper integrals**.

Definition 1.6.1. A function f is said to have an infinite discontinuity at c if, from the right or left,

$$\lim_{x\to c} f(x)=\infty \quad \text{or} \quad \lim_{x\to c} f(x)=-\infty$$

Definition 1.6.2. Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx$$

2. If f is continuous on the interval $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx=\lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3. If f is continuous on the interval $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx=\int_{-\infty}^c f(x)\,dx+\int_c^\infty f(x)\,dx \quad \text{where } c \text{ any real number.}$$

In the first two cases, the improper integral converges if the limit exists, otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 1.38. Evaluate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.

Solution 1.38.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln x]_1^b = \lim_{b\to\infty}(\ln b - 0) = \infty$$

Example 1.39. Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution 1.39.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b}-1\right) = 1$$

Example 1.40. Evaluate $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$.

Solution 1.40.

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = \lim_{b\to\infty}\left(\tan^{-1}b - 0\right) = \frac{\pi}{2}$$

Example 1.41. Evaluate $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$.

Solution 1.41. Use integration by parts, with $u = 1-x$ and $dv = e^{-x}dx$.

$$\begin{aligned}\int (1-x)e^{-x}\,dx &= -e^{-x}(1-x) - \int e^{-x}\,dx \\ &= \cancel{-e^{-x}} + xe^{-x} + \cancel{e^{-x}} + C = xe^{-x} + C\end{aligned}$$

Now, apply the definition of an improper integral.

$$\begin{aligned}\int_1^\infty (1-x)e^{-x}\,dx &= \lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx \\ &= \lim_{b\to\infty}\left[xe^{-x}\right]_1^b \\ &= \left(\lim_{b\to\infty}\frac{b}{e^b}\right) - \frac{1}{e} = -\frac{1}{e}\end{aligned}$$

Example 1.42. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$.

Solution 1.42. Note that the integrand is continuous on $(-\infty, \infty)$. To evaluate the integral, you can break it into two parts, choosing $c = 0$ as a convenient value, and using the substitution $u = e^x$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx &= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx \\
&= \lim_{b\to-\infty} \left[\tan^{-1}\left(e^x\right)\right]_b^0 + \lim_{b\to\infty} \left[\tan^{-1}\left(e^x\right)\right]_0^b \\
&= \lim_{b\to-\infty} \left[\frac{\pi}{4} - \tan^{-1}\left(e^b\right)\right] + \lim_{b\to\infty} \left[\tan^{-1}\left(e^b\right) - \frac{\pi}{4}\right]_0^b \\
&= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{2}
\end{aligned}$$

The second basic type of improper integral is one that has an infinite discontinuity at or between the limits of integration.

Definition 1.6.3. Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$, and has an infinite discontinuity at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

2. If f is continuous on the interval $(a, b]$, and has an infinite discontinuity at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

3. If f is continuous on the interval $[a, b]$, except for some $c \in (a, b)$ at which f has an infinite discontinuity, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In the first two cases, the improper integral converges if the limit exists, otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 1.43. Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$.

Solution 1.43. The integrand has an infinite discontinuity at $x = 0$. You can evaluate this integral as shown below.

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1 = \lim_{b\to 0^+}\frac{3}{2}\left[1-\sqrt[3]{b^2}\right] = \frac{3}{2}$$

Example 1.44. Evaluate $\displaystyle\int_0^2 \frac{1}{x^3}\,dx$.

Solution 1.44. Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\int_0^2 \frac{1}{x^3}\,dx = \lim_{b\to 0^+}\int_b^2 \frac{1}{x^3}\,dx = \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_b^2 = \lim_{b\to 0^+}\left[-\frac{1}{8}+\frac{1}{2b^2}\right] = \infty$$

Example 1.45. Evaluate $\displaystyle\int_{-1}^2 \frac{1}{x^3}\,dx$.

Solution 1.45. This integral is improper because the integrand has an infinite discontinuity at the interior point $x = 0$. So, you can write

$$\int_{-1}^2 \frac{1}{x^3}\,dx = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^2 \frac{1}{x^3}\,dx$$

From Example 1.44 you know that the second integral diverges. So, the original improper integral also diverges.

The integral in the next example is improper for two reasons. One limit of integration is infinite, and the integrand has an infinite discontinuity at the outer limit of integration.

Example 1.46. Evaluate $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$.

Solution 1.46. To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx &= \int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx \\ &= \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}(x+1)}\,dx + \lim_{c\to\infty}\int_1^c \frac{1}{\sqrt{x}(x+1)}\,dx \\ &= \lim_{b\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_b^1 + \lim_{c\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^c \\ &= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) \\ &= \pi\end{aligned}$$

This section concludes with a useful theorem describing the convergence or divergence of a common type of improper integral.

Theorem 1.6.1. ***A Special Type of Improper Integral***

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \textit{if} \quad p > 1 \\ \textit{diverges} & \textit{if} \quad p \le 1 \end{cases}$$

Exercise 1.6. Evaluate the following integrals.

(1) $\displaystyle\int_0^{\pi/4} \csc x\,dx$ (2) $\displaystyle\int_0^1 \frac{1}{3x-5}\,dx$ (3) $\displaystyle\int_0^1 x\ln x\,dx$ (4) $\displaystyle\int_5^\infty \frac{1}{x\sqrt{x^2-25}}\,dx$

1.7 Strategy for Integration

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Until now individual techniques have been applied in each section. For instance, we usually used substitution, integration by parts, and partial fractions. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter. Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived. Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Table of Integration Formulas Constants of integration have been omitted.

1. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

2. $$\int \frac{1}{x}\,dx = \ln|x|$$

3. $$\int e^x\,dx = e^x$$

4. $$\int a^x\,dx = \frac{a^x}{\ln a}$$

5. $$\int \sin x\,dx = -\cos x$$

6. $$\int \cos x\,dx = \sin x$$

7. $$\int \sec^2 x\,dx = \tan x$$

8. $$\int \csc^2 x\,dx = -\cot x$$

9. $$\int \sec x \tan x\,dx = \sec x$$

10. $$\int \csc x \cot x\,dx = -\csc x$$

11. $$\int \sec x\,dx = \ln|\sec x + \tan x|$$

12. $$\int \csc x\,dx = \ln|\csc x - \cot x|$$

13. $$\int \tan x\,dx = \ln|\sec x|$$

14. $$\int \cot x\,dx = \ln|\sin x|$$

15. $$\int \sinh x\,dx = \cosh x$$

16. $$\int \cosh x\,dx = \sinh x$$

17. $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

18. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$$

***19.** $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

***20.** $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

Once you are armed with these basic integration formulas, if you don't immediately see how to attack a given integral, you might try the following four-step strategy.

1. **Simplify the Integrand if Possible.** Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and

make the method of integration obvious. For example,

$$\int \sqrt{x}\left(1+\sqrt{x}\right)\,dx = \int \left(\sqrt{x}+x\right)\,dx$$

$$\text{and} \int (\sin x + \cos x)^2\,dx = \int \left(\sin^2 x + 2\cos x \sin x + \cos^2 x\right)\,dx$$
$$= \int [1+\sin(2x)]\,dx$$

2. **Look for an Obvious Substitution.** Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2-1}\,dx$$

we notice that if $u = x^2 - 1$, then $du = 2x dx$. Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. **Classify the Integrand According to Its Form.** If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

   a) **Trigonometric functions.** If is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or of $\cot x$ and $\csc x$, then we use the substitutions recommended in Section 1.3.

   b) **Rational functions.** If f is a rational function, we use the procedure of Section 1.5 involving partial fractions.

   c) **Integration by parts.** If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv according to the advice given in Section 1.2.

   d) **Radicals.** Particular kinds of substitutions are recommended when certain radicals appear.

      i. If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution according to the table in Section 1.4.

      ii. If $\sqrt[n]{ax+b}$ occurs, we use the rationalizing substitution $u = \sqrt[n]{ax+b}$. More generally, this sometimes works for $\sqrt[n]{g(x)}$.

4. **Try Again.** If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

   a) Try substitution. Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

   b) Try parts. Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions. Looking at Section 1.2, we see that it works on $\sin^{-1} x$.

   c) Manipulate the integrand. Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity.

   d) Relate the problem to previous problems. When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one.

   e) Use several methods. Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

Exercise 1.7. Evaluate the following integrals.

(1) $\displaystyle\int \left[\frac{\tan x}{\cos x}\right]^3 dx$ (2) $\displaystyle\int e^{\sqrt{x}}\, dx$ (3) $\displaystyle\int \frac{x^5+1}{x^3-3x^2+10}\, dx$

(4) $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\, dx$ (5) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\, dx$

CHAPTER 2

Infinite Series

Infinite series are sums of infinitely many terms. One of our aims in this chapter is to define exactly what is meant by an infinite sum. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in order to integrate such functions as e^{-x^2}, recall that we have previously been unable to do this. Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

2.1 Sequences

In mathematics, the word **sequence** is used in much the same way as in ordinary English. To say that a collection of objects or events is in sequence usually means that the collection is ordered so that it has an identified first member, second member, third member, and so on.

Mathematically, a sequence is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For example,

$$a_1, \quad a_2, \quad a_3, \quad \cdots, \quad a_n, \quad \cdots$$

The numbers $a_1, a_2, a_3, \cdots$ are the terms of the sequence. The number a_n is the nth term of the sequence, and the entire sequence is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 2.1. Listing the first few terms of the given sequences.

a) $a_n = 3 + (-1)^n$ b) $\left\{ \dfrac{n^2}{2^n - 1} \right\}$

Solution 2.1.

a) The terms of the sequence $a_n = 3 + (-1)^n$ are

$$\begin{array}{ccccccccc} n=1 & , & n=2 & , & n=3 & , & n=4 & , & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 2 & , & 4 & , & 2 & , & 4 & , & \cdots \end{array}$$

b) The terms of the sequence $\left\{ \dfrac{n^2}{2^n - 1} \right\}$ are

$$\begin{array}{ccccccccc} n=1 & , & n=2 & , & n=3 & , & n=4 & , & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & , & \frac{4}{3} & , & \frac{9}{7} & , & \frac{16}{15} & , & \cdots \end{array}$$

There are sequences that don't have a simple defining equation like the one in the next example.

Example 2.2. Find the terms of the **recursively** defined **Fibonacci sequence** f_n where $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

Solution 2.2. The terms of the sequence f_n are

$$\underbrace{\mathbf{1}, \quad \mathbf{1},}_{\textbf{Initial Terms}} \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad \cdots$$

Example 2.3. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \cdots\right\}$$

assuming that the pattern of the first few terms continues.

Solution 2.3. Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n+1$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms are alternately positive and negative, so we need to multiply by a power of $(-1)^n$. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore,

$$a_n = (-1)^{n-1}\,\frac{n+2}{5^n}$$

Definition 2.1.1. Definition of the Limit of a Sequence
A sequence $\{a_n\}$ has the limit ℓ and we write $\lim_{n\to\infty} a_n = \ell$ or $a_n \to \ell$ as $n \to \infty$ if we can make the terms a_n as close to ℓ as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Theorem 2.1.1. *Let ℓ be a real number. Let $f(x)$ be a function of a real variable such that $\lim_{x\to\infty} f(x) = \ell$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n then $\lim_{n\to\infty} a_n = \ell$.*

Example 2.4. Find the limit of the sequence whose nth term is $a_n = \left(1+\frac{1}{n}\right)^n$.

Solution 2.4. Previously, in Calculus 1, you learned that $\lim_{x\to\infty}\left(1+\frac{\alpha}{x}\right)^{\beta x} = e^{\alpha\beta}$. So, you can apply Theorem 2.1.1 to conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Example 2.5. Find the limit of the sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$

Solution 2.5. This is the sequence with general term $a_n = \dfrac{n-1}{n} = 1 - \dfrac{1}{n}$. Then by Theorem 2.1.1, $\lim\limits_{n\to\infty}\left[1-\dfrac{1}{n}\right] = 1$.

Example 2.6. Determine whether the sequence $\left\{\dfrac{n+\ln n}{n^2}\right\}$ converges or diverges.

Solution 2.6. Apply Theorem 2.1.1 directly on $a_n = \dfrac{n+\ln n}{n^2}$ to obtain using L'Hospital's Rule

$$\lim_{n\to\infty}\frac{n+\ln n}{n^2} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2n} = 0 \qquad \textbf{(Converges)}$$

Example 2.7. Determine whether the sequence $\left\{\dfrac{n^2(4n+1)(5n+3)}{6n^3+2}\right\}$ converges or diverges.

Solution 2.7. By Theorem 2.1.1, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2(4n+\cancel{1})(5n+\cancel{3})}{6n^3+\cancel{2}} &= \lim_{n\to\infty}\frac{n^2(4n)(5n)}{6n^3}\\ &= \lim_{n\to\infty}\frac{20\cancel{n^4}}{6\cancel{n^3}} = \lim_{n\to\infty}\frac{20n}{6} = \infty \qquad \textbf{(Diverges)}\end{aligned}$$

The following properties of limits of sequences parallel those given for limits of functions of a real variable in Calculus I.

Theorem 2.1.2. ***Properties of Limits of Sequences***
Let $\lim\limits_{n\to\infty} a_n = A$ *and* $\lim\limits_{n\to\infty} b_n = B$. *Then*

(1) $\lim\limits_{n\to\infty}[a_n \pm b_n] = A \pm B$ *(2)* $\lim\limits_{n\to\infty}[c\,a_n] = c\,A$

(3) $\lim\limits_{n\to\infty}[a_n\,b_n] = A\,B$ *(4)* $\lim\limits_{n\to\infty}\left[\dfrac{a_n}{b_n}\right] = \dfrac{A}{B}$ *if* $b_n, B \neq 0$

Example 2.8. Determine whether the sequence $\left\{\dfrac{2-3e^{-n}}{6+4e^{-n}}\right\}$ converges or diverges.

Solution 2.8. Observe that $\lim_{n\to\infty}\left[2-3e^{-n}\right]=2$ and $\lim_{n\to\infty}\left[6+4e^{-n}\right]=6$. According to Theorem 2.1.2, we have

$$\lim_{n\to\infty}\frac{2-3e^{-n}}{6+4e^{-n}}=\frac{2}{6}=\frac{1}{3} \qquad \textbf{(Converges)}$$

Theorem 2.1.3. ***Sequences of the Forms r^n and $\frac{1}{n^r}$.***

1. *Suppose r is a nonzero constant. The sequence $\{r^n\}$ converges to 0 if $|r|<1$ and diverges if $|r|>1$.*
2. *The sequence $\frac{1}{n^r}$ converges to 0 for r any positive rational number.*

Example 2.9. Determine the convergence or divergence of the sequence with the given nth term.

(1) $a_n=e^{-n}$ (2) $a_n=\left(\frac{3}{2}\right)^n$ (3) $a_n=\frac{4}{\sqrt{n^5}}$

Solution 2.9. By Theorem 2.1.3,

1. since $a_n=e^{-n}=\left(\frac{1}{e}\right)^n$ and $|r|=\frac{1}{e}<1$, then $a_n=e^{-n}$ converges to 0.
2. since $|r|=\frac{3}{2}>1$, then the sequence $a_n=\left(\frac{3}{2}\right)^n$ diverges.
3. the sequence $a_n=\frac{4}{\sqrt{n^5}}=\frac{4}{n^{\frac{5}{2}}}$ converges to 0 since $r=\frac{5}{2}$ is positive rational number.

Theorem 2.1.4. ***Squeeze Theorem for Sequences***
If

$$\lim_{n\to\infty}a_n=\ell=\lim_{n\to\infty}b_n$$

and there exists an integer N such that $a_n\le c_n\le b_n$ for all $n\ge N$, then

$$\lim_{n\to\infty}c_n=\ell$$

Example 2.10. Show that the sequence $\left\{\frac{\cos n}{n^2}\right\}$ converges.

Solution 2.10. We have,

$$-\frac{1}{n^2} \le \frac{\cos n}{n^2} \le \frac{1}{n^2}$$

Since both $\left\{-\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n^2}\right\}$ tend to 0, then by Theorem 2.1.4, $\left\{\frac{\cos n}{n^2}\right\}$ converges to 0.

Example 2.11. Determine whether the sequence $\left\{\frac{2^n}{n!}\right\}$ converges or diverges.

Solution 2.11. Even though $\lim_{n\to\infty}\left(\frac{2^n}{n!}\right) = \frac{\infty}{\infty}$ we can not use L'Hospital's Rule since we have studied no function $f(x) = x!$. We can use Theorem 2.1.4 as follows.

$$0 \le \frac{2^n}{n!} = \frac{\overbrace{2\cdot 2\cdot 2\cdots 2\cdot 2\cdot 2}^{n \textbf{ factors of } 2}}{\underbrace{1\cdot 2\cdot 3\cdots(n-2)\cdot(n-1)\cdot n}_{n \textbf{ factors}}} = \overbrace{\frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdots\frac{2}{n-2}\cdot\frac{2}{n-1}\cdot\frac{2}{n}}^{n \textbf{ fractions}}$$

$$\le 2\cdot 1\cdot \underbrace{\frac{2}{3}\cdot\frac{2}{3}\cdots\frac{2}{3}\cdot\frac{2}{3}}_{n-2 \textbf{ fractions}} = 2\left(\frac{2}{3}\right)^{n-2} = \frac{9}{2}\left(\frac{2}{3}\right)^n$$

Since $\lim_{n\to\infty}\left[\frac{9}{2}\left(\frac{2}{3}\right)^n\right] = 0$ by Theorem 2.1.3, then by Theorem 2.1.4 the sequence $\left\{\frac{2^n}{n!}\right\}$ converges to 0.

Theorem 2.1.5. ***Absolute Value Theorem***
For the sequence $\{a_n\}$, *if* $\lim_{n\to\infty}|a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Example 2.12. Determine whether the sequence $\left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges or diverges.

Solution 2.12. Since $\lim_{n\to\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\frac{1}{\sqrt{n}}\right] = 0$, then by Theorem 2.1.5, the sequence $\left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges to 0.

Example 2.13. Determine whether the sequence $a_n = (-1)^n$ converges or diverges.

Solution 2.13. If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, \cdots\}$$

Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim\limits_{n\to\infty}\left[(-1)^n\right]$ does not exist; that is, the sequence $a_n = (-1)^n$ is divergent.

Theorem 2.1.6. *If $f(x)$ is continuous and the limit $\lim\limits_{n\to\infty} a_n = \ell$ exists, then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(\ell).$$

Example 2.14. Find $\lim\limits_{n\to\infty} \sin\left(\dfrac{\pi}{n}\right)$.

Solution 2.14. Because the sine function is continuous at 0, Theorem 2.1.6 enables us to write

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0.$$

Example 2.15. Show that the sequence $\left\{(1+n)^{\frac{1}{n}}\right\}$ converges.

Solution 2.15. By Theorem 2.1.1 and Theorem 2.1.6,

$$\begin{aligned}
\lim_{n\to\infty} (1+n)^{\frac{1}{n}} &= \lim_{n\to\infty} e^{\ln\left[(1+n)^{\frac{1}{n}}\right]} \\
&= e^{\lim\limits_{n\to\infty} \ln\left[(1+n)^{\frac{1}{n}}\right]} = e^{\lim\limits_{n\to\infty} \frac{\ln(1+n)}{n}} \\
&= e^{\lim\limits_{n\to\infty} \frac{1}{1+n}} = e^0 = 1
\end{aligned}$$

Theorem 2.1.7. *If $\{a_n\}$ converges to ℓ, then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = \ell$.*

Example 2.16. Assuming that the sequence defined by the recurrence relation

$$a_1 = 2, \qquad a_{n+1} = \frac{6 + a_n}{2}, \qquad \text{for } n = 1, 2, 3, \cdots$$

converges, show that the limit is 6.

Solution 2.16. Since the sequence $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n = \ell$ exists. Theorem 2.1.7 does not tell us what the value of the limit is. But we can use the given recurrence relation to write

$$\ell = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[\frac{6+a_n}{2}\right] = \frac{6+\ell}{2} \implies \ell = 6$$

The next theorem gives some limits that arise frequently.

Theorem 2.1.8. *The following sequences converge to the limits listed below:*

(1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ *(2)* $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(3) $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$ *(4)* $\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad (x > 0)$

(5) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x \quad (\textit{any } x)$ *(6)* $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\textit{any } x)$

Exercise 2.1.

1. Determine whether the sequence converges or diverges.

(1) $\dfrac{11-2e^n}{3e^n}$ (2) $\cos(n\pi)$ (3) $\dfrac{2^n}{3^n+1}$

(4) $n^{2/(n+1)}$ (5) $\sin(n\pi)$ (6) $\left(-\dfrac{1}{3}\right)^n$

(7) n^3e^{-n} (8) $\dfrac{\sin^2 n}{4^n}$ (9) $\sin\left(\dfrac{n\pi}{2}\right)$

(10) $\ln\left(\dfrac{4n+1}{3n-1}\right)$ (11) $n\sin\left(\dfrac{6}{n}\right)$ (12) $\dfrac{n!}{n^n}$

(13) $\dfrac{e^n-e^{-n}}{e^n+e^{-n}}$ (14) $\dfrac{(-1)^n}{n^2+1}$ (15) $(-1)^n\dfrac{2n^3}{n^3+1}$

(16) $\left(\dfrac{n+3}{n+1}\right)^n$ (17) $\left(1-\dfrac{2}{n}\right)^n$ (18) $\sqrt{n+1}-\sqrt{n}$

(19) $\dfrac{1+(-1)^n}{n}$ (20) $\dfrac{(n-2)!}{n!}$ (21) $\left(1+\dfrac{1}{n^2}\right)^n$

(22) $\left(2+\frac{4}{n^2}\right)^{\frac{1}{3}}$ (23) $\frac{n}{n+\sqrt[n]{n}}$ (24) $\frac{e^n+3^n}{5^n}$

(25) $\frac{3-4^n}{2+7\cdot 3^n}$ (26) $\tan^{-1}\left(\frac{n^2}{n+1}\right)$ (27) $\tan\left(\frac{2n\pi}{1+8n}\right)$

(28) $\sqrt{\frac{n+1}{9n+1}}$ (29) $\left(\frac{3}{n}\right)^{\frac{1}{n}}$ (30) $\left[2-\frac{1}{3^n}\right]\left[3+\frac{1}{2^n}\right]$

(31) $\frac{1}{n}\int_1^n \frac{1}{x}\,dx$ (32) $\frac{1}{3}, \frac{-1}{9}, \frac{1}{27}, \frac{-1}{81}, \cdots$ (33) $2\ln(3n)-\ln\left(1+n^2\right)$

2. The recursively defined sequence $a_{n+1}=\frac{1}{2}\left[a_n+\frac{5}{a_n}\right]$ is know to converge to a given initial value $a_1>0$. Find the limit of the sequence. $\left[\textbf{Hint: } \lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty} a_n=\ell\right]$

3. For what positive values of b does the following sequence converge?

$$b, 0, b^2, 0, b^3, 0, b^4, \cdots$$

$\left[\textbf{Hint: When } \lim_{n\to\infty} b^n=0?\right]$.

4. Evaluate $\lim_{n\to\infty}\sqrt[n]{2^n+3^n}$. $\left[\textbf{Hint: } \text{Show that } 3\le\sqrt[n]{2^n+3^n}\le 3\sqrt[n]{2}\right]$.

5. Give an example of a divergent sequence $\{a_n\}$ such that $\{|a_n|\}$ converges. [**Hint:** See Exercise 3.1.1.15].

6. Show, by giving an example, that there exists divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n+b_n\}$ converges. $\left[\textbf{Hint: } \text{What about } n^2, \frac{1}{n}-n^2?\right]$.

7. Determine whether the sequence defined as follows is convergent or divergent.

$$a_1=1 \qquad a_{n+1}=4-a_n \quad \text{for } n\ge 1$$

What happens if the first term is $a_1=2$? [**Hint:** Write the first few elements of the sequence].

2.2 Series and Convergence

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representations of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333\cdots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

which suggests that the decimal representation of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

One important application of infinite sequences is in representing **infinite summations**. Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is an **infinite series** (or simply a **series**). The numbers $a_1, a_2, a_3, \cdots$ are the terms of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\sum a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following sequence of **partial sums**.

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n
\end{aligned}$$

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

Definition 2.2.1. Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the nth partial sum is given by

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to S then the series $\sum_{n=1}^{\infty} a_n$ **converges** to S. If $\{S_n\}$ diverges, then the series **diverges**.

Example 2.17. Determine whether the following series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Solution 2.17. The partial sums are

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 - 1 = 0 \\ S_3 &= 1 - 1 + 1 = 1 \\ S_n &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

and so forth. Thus, the sequence of partial sums is

$$1, 0, 1, 0, 1, 0, \cdots$$

Since this is a divergent sequence, the given series diverges and consequently has no sum.

Telescoping Sums

A **telescoping series** is a series whose partial sums eventually only have a fixed number of terms after cancellation. Such a technique is also known as the **method of differences**. The next example treats a convergent telescoping series, where the partial sums are particularly easy to evaluate.

Example 2.18. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(1+n)}$ converges or diverges. If it converges, find the sum.

Solution 2.18. We will begin by rewriting S_n in closed form. This can be accomplished by using the method of partial fractions to obtain (verify)

$$\frac{1}{n(1+n)} = \frac{1}{n} - \frac{1}{1+n}$$

from which we obtain the sum

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{1+k}\right] \\
&= \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{1+n}\right] \\
&= 1 - \frac{1}{1+n}
\end{aligned}
$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+n)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{1+n}\right] = 1.$

Example 2.19. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converges or diverges. If it converges, find the sum.

Solution 2.19. We will begin by rewriting S_n in closed form by writing

$$\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$$

from which we obtain the sum

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right) \\
&= [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + \cdots + [\ln(n) - \ln(n+1)] \\
&= -\ln(n+1)
\end{aligned}
$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} [-\ln(1+n)] = -\infty$. Hence, the sum diverges.

Geometric Series

In many important series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is a and each term is obtained by multiplying the preceding term by r, then the series has the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

Such series are called **geometric series**, and the number r is called the **ratio** for the series.

Theorem 2.2.1. ***Convergence of a Geometric Series***
A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$ then the series converges to the sum $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Example 2.20. Determine whether the series $\sum_{n=0}^{\infty} \frac{5}{4^n}$ converges, and if so find its sum.

Solution 2.20. This is a geometric series with $a = 5$ and $r = \frac{1}{4}$. Since $|r| < 1$, the series converges and the sum is $\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$.

Example 2.21. Determine whether the series $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ converges, and if so find its sum.

Solution 2.21. This is a geometric series with $a = 1$ and $r = \frac{5}{4}$. Since $|r| > 1$, the series diverges.

Example 2.22. Determine whether the series $\sum_{n=1}^{\infty} \left(2^{2n}5^{1-n}\right)$ converges, and if so find its sum.

Solution 2.22. This is a geometric series in concealed form, since we can rewrite it as

$$\sum_{n=1}^{\infty} \left(2^{2n}5^{1-n}\right) = \sum_{n=1}^{\infty} \frac{4^n \cdot 5}{5^n} = \sum_{n=1}^{\infty} 5\left(\frac{4}{5}\right)^n$$

with $a = 4$ and $r = \frac{4}{5}$. Since $|r| < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = 20$$

Example 2.23. Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution 2.23. We can write

$$\begin{aligned} 0.\overline{08} &= 0.08 + 0.0008 + 0.000008 + \cdots \\ &= \frac{8}{100} + \frac{8}{100^2} + \frac{8}{100^3} + \cdots = \sum_{n=1}^{\infty} \frac{8}{100^n} \end{aligned}$$

So the given decimal is the sum of a geometric series with $a = \dfrac{8}{100}$ and $r = \dfrac{1}{100}$. Thus,

$$0.\overline{08} = \sum_{n=1}^{\infty} \frac{8}{100^n} = \frac{\frac{8}{100}}{1-\frac{1}{100}} = \frac{8}{99}$$

Example 2.24. Find all values of x for which the series $\displaystyle\sum_{n=0}^{\infty} 3\left[-\frac{x}{2}\right]^n$ converges, and find the sum of the series for those values of x.

Solution 2.24. This is a geometric series with $a = 3$ and $r = -\dfrac{x}{2}$. It converges if $\left|-\dfrac{x}{2}\right| < 1$, or equivalently, when $|x| < 2$. When the series converges its sum is

$$\sum_{n=0}^{\infty} 3\left[-\frac{x}{2}\right]^n = \frac{3}{1+\frac{x}{2}} = \frac{6}{2+x}$$

The following properties are direct consequences of the corresponding properties of limits of sequences.

Theorem 2.2.2. ***Properties of Infinite Series***
Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums.

1. $\sum c a_n = cA$
2. $\sum (a_n \pm b_n) = A \pm B$

Example 2.25. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left[\frac{3}{4^n} - \frac{2}{5^{n-1}}\right]$.

Solution 2.25. The series $\displaystyle\sum_{n=1}^{\infty}\frac{3}{4^n}$ is a convergent geometric series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$, and the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{5^{n-1}}$ is also a convergent geometric series $\left(a = 2, r = \frac{1}{5}\right)$. Thus, from Theorems 2.2.2 the given series converges and

$$\sum_{n=1}^{\infty}\left[\frac{3}{4^n} - \frac{2}{5^{n-1}}\right] = \sum_{n=1}^{\infty}\frac{3}{4^n} - \sum_{n=1}^{\infty}\frac{2}{5^{n-1}} = \frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{2}{1-\frac{1}{5}} = -\frac{3}{2}$$

The following theorem states that if a series converges, the limit of its term must be 0.

Theorem 2.2.3. ***Limit of the nth Term of a Convergent Series***
If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The contrapositive of Theorem 2.2.3 provides a useful test for divergence. This **nth-Term Test for Divergence** states that if the limit of the term of a series does not converge to 0, the series must diverge.

Theorem 2.2.4. ***nth-Term Test for Divergence***
If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges .

Example 2.26. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{1+n}$ converges or diverges.

Solution 2.26. Since $\lim_{n\to\infty} \frac{n}{1+n} = 1 \neq 0$ then the series diverges.

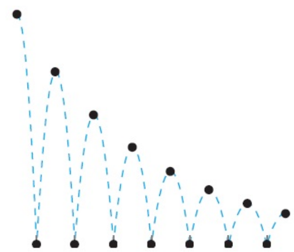
Example 2.27. Determine whether the series $\sum_{n=0}^{\infty} 2^n$ converges or diverges.

Solution 2.27. Since $\lim_{n\to\infty} 2^n = \infty \neq 0$ then the series diverges.

Example 2.28. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Solution 2.28. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ then the nth-Term Test for Divergence does not apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

Example 2.29. A ball is dropped from a height of 6 feet and begins bouncing, as shown in the figure below. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance travelled by the ball.



Solution 2.29. When the ball hits the ground for the first time, it has travelled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance travelled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = \underbrace{6\left(\frac{3}{4}\right)}_{\textbf{up}} + \underbrace{6\left(\frac{3}{4}\right)}_{\textbf{down}} = 12\left(\frac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\textbf{up}} + \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\textbf{down}} = 12\left(\frac{3}{4}\right)^2$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned} D &= 6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \cdots \\ &= 6 + 12\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = 6 + 12\left[\frac{\frac{3}{4}}{1-\frac{3}{4}}\right] = 6 + 12 \times 3 = 42 \text{ feet.} \end{aligned}$$

Exercise 2.2.

1. Find the sum of the series if it converges.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12}$ (2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}}$ (3) $\displaystyle\sum_{n=0}^{\infty} 5^n 4^{-n}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n-1}{4^n}$ (5) $\displaystyle\sum_{n=1}^{\infty} 10$ (6) $\displaystyle\sum_{n=1}^{\infty} \ln\left[\frac{n}{3n+1}\right]$

(7) $\sum_{n=1}^{\infty} \frac{4^{n+2}}{7^{n-1}}$ (8) $\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+4^n}$ (9) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$

(10) $\sum_{n=1}^{\infty} \tan^{-1} n$ (11) $\sum_{n=1}^{\infty} \cos\left[\frac{1}{n}\right]$ (12) $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$

(13) $\sum_{n=1}^{\infty} \ln\left[\frac{1}{3^n}\right]$ (14) $\sum_{n=1}^{\infty} \left[1-\frac{1}{n}\right]^n$ (15) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

2. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $S_n = \frac{n-1}{n+1}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

3. Let $a_n = \frac{2n}{3n+1}$. Determine whether the sequence $\{a_n\}$ and the series $\sum a_n$ are convergent?

4. A sequence of terms is defined by $a_n = (5-n)a_{n-1}$ where $a_1 = 1$. Find $\sum_{n=1}^{\infty} a_n$.

5. Show that : $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1$.

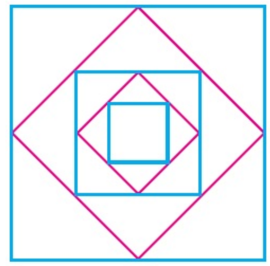
6. Write the repeating decimal number $1.\overline{314}$ as a quotient of integers.

7. Determine the values of x for which the series $\sum_{n=0}^{\infty} 2^n x^{2n}$ converges.

8. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.

9. Determine whether the series $\frac{1}{1.1} + \frac{1}{1.11} + \frac{1}{1.111} + \cdots$ converges? $\left[\textbf{Hint: } a_n = \frac{1}{\sum_{k=0}^{n}\left(\frac{1}{10^k}\right)}\right]$

10. Find the sum of the series $\frac{1+9}{25} + \frac{1+27}{125} + \frac{1+81}{625} \cdots$.



11. Show that for all real values of x,

$$\sin x - \frac{1}{2}\sin^2 x + \frac{1}{4}\sin^3 x - \frac{1}{8}\sin^4 x + \cdots = \frac{2\sin x}{2+\sin x}$$

12. Find the value of c for which the series equals the indicated sum.

$$\sum_{n=2}^{\infty}(1+c)^{-n} = 2$$

2.3 The Integral Test and $p-$Series

In this and the following section, you will study several convergence tests that apply to series with **positive** terms.

Theorem 2.3.1. ***The Integral Test***
If f is positive, continuous, and decreasing for $x \geq N$ and $a_n = f(n)$, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

Example 2.30. Apply the Integral Test to the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$.

Solution 2.30. The function $f(x) = \frac{x}{x^2+1}$ is positive and continuous for $x \geq 1$. To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{\left(x^2+1\right)(1)-(x)(2x)}{\left(x^2+1\right)^2} = \frac{-x^2+1}{\left(x^2+1\right)^2} < 0 \quad \text{for} \quad x > 1$$

It follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\int_1^{b} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\left[\ln\left(x^2+1\right)\right]_1^b = \frac{1}{2}\lim_{b\to\infty}\left[\ln\left(b^2+1\right)-\ln(2)\right]=\infty\end{aligned}$$

So, the series diverges.

Example 2.31. Apply the Integral Test to the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$.

Solution 2.31. The function $f(x) = \dfrac{1}{x^2+1}$ is positive and continuous for $x \geq 1$. To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{-2x}{\left(x^2+1\right)^2} < 0 \quad \text{for} \quad x > 1$$

It follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1}x\right]_1^b \\ &= \lim_{b\to\infty}\left[\tan^{-1}b-\tan^{-1}1\right] = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}\end{aligned}$$

So, the series converges.

Example 2.32. Determine whether the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$ converges or diverges.

Solution 2.32. The function $f(x) = \dfrac{1}{x\ln x}$ is positive and continuous for $x \geq 2$. To determine whether f is decreasing, first rewrite f as $f(x) = (x\ln x)^{-1}$ and then find its derivative.

$$f'(x) = (-1)(x\ln x)^{-2}(1+\ln x) = -\frac{1+\ln x}{x^2(\ln x)^2} < 0 \quad \text{for} \quad x > 2$$

It follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \int_2^\infty \frac{1/x}{\ln x}\,dx = \lim_{b\to\infty}\left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty}\left[\ln(\ln b) - \ln(\ln 2)\right] = \infty$$

So, the series diverges.

In the remainder of this section, you will investigate a second type of series that has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a $p-$**series**, where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is the **harmonic** series. A general harmonic series is of the form $\sum \frac{1}{an+b}$. The Integral Test is convenient for establishing the convergence or divergence of $p-$series.

Theorem 2.3.2. *Convergence of $p-$Series*
The $p-$series $\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Example 2.33. Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$ converges or diverges.

Solution 2.33. The series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{3}}}$ diverges since it is a $p-$series with $p = \frac{1}{3} < 1$.

Exercise 2.3.

1. Determine whether the series converges or diverges.

(1) $\sum_{n=1}^{\infty}\frac{1}{n+3}$ (2) $\sum_{n=1}^{\infty}3^{-n}$ (3) $\sum_{n=1}^{\infty}ne^{-\frac{n}{2}}$

(4) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ (5) $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ (6) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

(7) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ (8) $\sum_{n=1}^{\infty} \frac{2}{e^n+e^{-n}}$ (9) $\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n^5}}$

(10) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

2. Find the sum of the series $\sum_{n=2}^{\infty} \ln\left[1-\frac{1}{n^2}\right]$. $\left[\,\textbf{Hint:} \sum = -\ln 2\right]$.

3. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.
$\left[\,\textbf{Hint:}\ b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b}\right]$

2.4 Comparisons of Series

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test non-applicable. For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \text{ is geometric, but } \sum_{n=0}^{\infty} \frac{n}{2^n} \text{ is not.}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ is a } p\text{-series, but } \sum_{n=1}^{\infty} \frac{1}{n^3+1} \text{ is not.}$$
$$\frac{n}{\left(n^2+3\right)^2} \text{ is easily integrated, but } \frac{n^2}{\left(n^2+3\right)^2} \text{ is not.}$$

In this section you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to compare a series having complicated terms with a simpler series whose convergence or divergence is known.

Theorem 2.4.1. ***Direct Comparison Test***
Let $0 < a_n \le b_n$ *for all* n.

1. *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Example 2.34. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}3^n}$.

Solution 2.34. This series resembles $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is convergent geometric series. Term-by-term comparison yields

$$a_n = \frac{1}{2+\sqrt{n}3^n} \le \frac{1}{3^n} = b_n \quad \text{for} \quad n \ge 1$$

So, by the Direct Comparison Test, the series converges.

Example 2.35. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

Solution 2.35. This series resembles $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ which is divergent $p-$series. Term-by-term comparison yields

$$\frac{1}{2+\sqrt{n}} \le \frac{1}{\sqrt{n}} \quad \text{for} \quad n \ge 1$$

which does not meet the requirements for divergence. (Remember that if term-by-term comparison reveals a series that is smaller than a divergent series, the Direct Comparison Test tells you nothing.) Still expecting the series to diverge, you can compare the given series with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent harmonic series. In this case, term-by-term comparison yields

$$a_n = \frac{1}{n} \le \frac{1}{2+\sqrt{n}} = b_n \quad \text{for} \quad n \ge 4$$

and, by the Direct Comparison Test, the given series diverges.

Example 2.36. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$.

Solution 2.36. This series resembles $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent $p-$series. Term-by-term comparison yields

$$a_n = \frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2} = b_n \quad \text{for} \quad n \geq 1$$

So, by the Direct Comparison Test, the series converges.

Example 2.37. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{\sqrt{n^6+5n^3+6}}$.

Solution 2.37. This series resembles $\sum_{n=1}^{\infty} \frac{\pi/2}{n^3}$ which is convergent $p-$series. Term-by-term comparison yields

$$a_n = \frac{\tan^{-1} n}{\sqrt{n^6+5n^3+6}} \leq \frac{\pi/2}{\sqrt{n^6}} \leq \frac{\pi/2}{n^3} = b_n \quad \text{for} \quad n \geq 1$$

So, by the Direct Comparison Test, the series converges.

Example 2.38. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution 2.38. Because $\ln n$ grows more slowly than n^c for any positive constant c, we can compare the series to a convergent $p-$series by choosing $c > 0$ such that

$$\frac{3}{2} - c > 1 \implies 0 < c < \frac{1}{2}$$

To get the $p-$series, we see that

$$a_n = \frac{\ln n}{n^{3/2}} \leq \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}} = b_n \quad \text{for} \quad n \geq 1$$

Since $\frac{1}{n^{5/4}}$ is a convergent $p-$series, then by the Direct Comparison Test, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converges.

Often a given series closely resembles a series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the Limit Comparison Test.

Theorem 2.4.2. ***Limit Comparison Test***
Suppose that $a_n > 0$, $b_n > 0$, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \ell$ where ℓ is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 2.39. Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a>0, \quad b>0.$$

Solution 2.39. By comparison with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ you have

$$\lim_{n\to\infty} \frac{1/(an+b)}{1/n} = \lim_{n\to\infty} \frac{n}{an+b} = \frac{1}{a}$$

Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the given series diverges.

The Limit Comparison Test works well for comparing a messy algebraic series with a $p-$series. In choosing an appropriate $p-$series, you must choose one with an nth term of the same magnitude as the nth term of the given series. In other words, when choosing a series for comparison, you can disregard all but the highest powers of in both the numerator and the denominator.

Example 2.40. Determine the convergence or divergence of $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+1}$.

Solution 2.40. Disregarding all but the highest powers of in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \text{convergent } p-\text{series.}$$

Because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

you can conclude by the Limit Comparison Test that the given series converges.

Example 2.41. Determine the convergence or divergence of $\sum\limits_{n=1}^{\infty} \dfrac{n2^n}{1+4n^3}$.

Solution 2.41. A reasonable comparison would be with the divergent series $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{n^2}$. Note that this series diverges by the nth-Term Test. From the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n2^n}{1+4n^3}\right)\left(\frac{n^2}{2^n}\right) = \lim_{n\to\infty} \frac{n^3}{4n^3+1} = \frac{1}{4}$$

you can conclude that the given series diverges.

Exercise 2.4.

1. Determine whether the series converges or diverges.

   (1) $\sum\limits_{n=3}^{\infty} \dfrac{\ln n}{n^5}$ (2) $\sum\limits_{n=2}^{\infty} \dfrac{1+2n}{n\ln n}$ (3) $\sum\limits_{n=1}^{\infty} \dfrac{2+\sin n}{\sqrt[3]{n^4+1}}$

   (4) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ (5) $\sum\limits_{n=1}^{\infty} \dfrac{n^2-n+2}{3n^5+n^2}$ (6) $\sum\limits_{n=2}^{\infty} \dfrac{n}{(4n+1)^{3/2}}$

   (7) $\sum\limits_{n=1}^{\infty} \sin\left(\dfrac{1}{n}\right)$ (8) $\sum\limits_{n=2}^{\infty} \dfrac{1}{n!}$ (9) $\sum\limits_{n=1}^{\infty} \dfrac{1}{4\sqrt[3]{n}-1}$

2. Show that if the series $\sum a_n$ of positive terms converges, then $\sum \ln(1+a_n)$ converges.

3. The meaning of decimal representation of a number $0.d_1d_2d_3\cdots$ is that

   $$0.d_1d_2d_3\cdots = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots$$

   Show that this series always converges. $\left[\textbf{Hint:}\ \dfrac{d_i}{10^i} \le \dfrac{9}{10^i}\right]$.

2.5 Alternating Series

So far, most series you have dealt with have had positive terms. In this section and the following section, you will study series that contain both positive and negative terms. The simplest such series is an **alternating series**, whose terms alternate in sign.

Theorem 2.5.1. ***Alternating Series Test***
Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ *for all* n.

It is not essential for condition (2) in Theorem 2.5.1 to hold for all terms; an alternating series will converge if condition (1) is true and condition (2) holds eventually. If an alternating series violates condition (1) of the alternating series test, then the series must diverge by the nth-Term Test.

Example 2.42. Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Solution 2.42. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 2.5.1 is satisfied. Also note that the second condition of Theorem 2.5.1 is satisfied because

$$\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \le 1 \quad \text{for all } n \implies a_{n+1} \le a_n$$

So, applying the Alternating Series Test, you can conclude that the series converges.

Example 2.43. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

Solution 2.43. To apply the Alternating Series Test, note that, for $n \ge 1$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^n}{n/2^{n-1}} = \frac{n+1}{2n} \le 1 \implies a_{n+1} \le a_n.$$

Furthermore, by L'Hopital's Rule,

$$\lim_{n\to\infty} \frac{n}{2^{n-1}} = \lim_{n\to\infty} \frac{1}{2^{n-1} \ln 2} = 0$$

Therefore, by the Alternating Series Test, the series converges.

Example 2.44. Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{3n-1}$.

Solution 2.44. The series diverges by the nth-Term Test since

$$\lim_{n\to\infty}\frac{2n+1}{3n-1}=\frac{2}{3}\neq 0.$$

Example 2.45. Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+1}$.

Solution 2.45. In order to show that the terms of the series satisfy the condition $a_{n+1}\leq a_n$, let us consider the function $f(x)=\frac{\sqrt{x}}{x+1}$ for which $a_n=f(n)$. From the derivative we see that

$$f'(x)=-\frac{x-1}{2\sqrt{x}(x+1)^2}<0 \quad \text{for} \quad x>1$$

and hence, the function $f(x)$ decreases for $x>1$. Thus, $a_{n+1}\leq a_n$ is true for $n\geq 1$. Moreover, $\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}=\lim_{n\to\infty}\frac{1}{2\sqrt{n}}=0$. Therefore, by the Alternating Series Test, the series converges.

Example 2.46. Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$.

Solution 2.46. Note that $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which is convergent as shown in Example 2.42.

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty}|a_n|=|a_1|+|a_2|+|a_3|+\cdots$$

whose terms are the absolute values of the terms of the original series.

Definition 2.5.1. Absolute Convergence
A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum|a_n|$ is convergent.

Example 2.47. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent or not.

Solution 2.47. The series is absolutely convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p-$series.

The next theorem shows that absolute convergence implies convergence.

Theorem 2.5.2. ***Absolute Convergence Test***
If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

The converse of Theorem 2.5.2 is not true. For instance, the **alternating harmonic series** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional**.

Definition 2.5.2. Conditional Convergence
A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 2.48. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent, divergent or conditionally convergent series.

Solution 2.48. The given series can be shown to be convergent by the Alternating Series Test. Moreover, because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent $p-$series, the given series is conditionally convergent.

Example 2.49. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{3^n}$ is convergent, divergent or conditionally convergent series.

Solution 2.49. This is not an alternating series. However, because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{\frac{n(n+1)}{2}}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent geometric series, you can apply Theorem 2.5.2 to conclude that the given series is absolutely convergent, and therefore convergent.

Example 2.50. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{n!}{2^n}$ is convergent, divergent or conditionally convergent series.

Solution 2.50. By the nth-Term Test for Divergence, you can conclude that this series diverges.

Exercise 2.5.

1. Determine whether the series is convergent, divergent or conditionally convergent.

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3n+1}$ (2) $\sum_{n=1}^{\infty}\left[\frac{-1}{e}\right]^n$ (3) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\ln(1+n)}$

(4) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+\sqrt{n}}{n+1}$ (5) $\sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1}n}{n^2}$ (6) $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{(2n-1)\pi}{2}\right]$

2. Explain why the following series converges for every positive value of x.

$$e^{-x}\sin(x)+e^{-2x}\sin(2x)+e^{-3x}\sin(3x)+\cdots$$

$$\left[\textbf{Hint:}\ \text{Show that } \sum_{n=1}^{\infty}\left|e^{-nx}\sin(nx)\right| \text{ converges.}\right]$$

2.6 The Ratio and Root Tests

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult. In such cases the next tests can often be used, since it works exclusively with the terms of the given series, it requires neither an initial guess about convergence nor the discovery of a series for comparison.

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio $\frac{a_{n+1}}{a_n}$. For a geometric series $\sum ar^n$, this rate is a constant r and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

Theorem 2.6.1. ***Ratio Test***
Let $\sum a_n$ be a series with non-zero terms.

1. *$\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.*
2. *$\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.*
3. *The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.*

Example 2.51. Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{2^n}{n!}$.

Solution 2.51. Because $a_n = \frac{2^n}{n!}$, you can write the following.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!}\div\frac{2^n}{n!}\right] = \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!}\times\frac{n!}{2^n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{2\cdot\cancel{2^n}}{(n+1)\cancel{n!}}\times\frac{\cancel{n!}}{\cancel{2^n}}\right] = \lim_{n\to\infty}\left[\frac{2}{n+1}\right] = 0 < 1$$

$\therefore$ This series converges.

Example 2.52. Determine whether the series $\sum_{n=1}^{\infty}\frac{n^2 2^{n+1}}{3^n}$ converges or diverges.

Solution 2.52. This series converges because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[(n+1)^2\left(\frac{2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2 2^{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\frac{2(n+1)^2}{3n^2} = \frac{2}{3} < 1$$

Example 2.53. Determine whether the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ converges or diverges.

Solution 2.53. This series diverges because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{n!}{n^n}\right)\right] = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{n+1}\left(\frac{1}{n^n}\right)\right]$$
$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left[\frac{n+1}{n}\right]^n = \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n = e > 1$$

Example 2.54. Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$.

Solution 2.54. Here, the Ratio Test is inconclusive because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right)\right] = \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)\right] = 1$$

To determine whether the series converges, you need to try a different test. In this case, you can apply the Alternating Series Test to show that this series is convergent. See Example 2.45.

The next test for convergence or divergence of series works especially well for series involving nth powers.

Theorem 2.6.2. ***Root Test***
Let $\sum a_n$ be a series.

1. *$\sum a_n$ converges absolutely if $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$.*
2. *$\sum a_n$ diverges if $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$.*
3. *The Root Test is inconclusive if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$.*

Note 2.1. The Root Test is always inconclusive for any $p-$series.

Example 2.55. Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{e^{2n}}{n^n}$.

Solution 2.55. The series converges, since

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty}\frac{e^2}{n} = 0 < 1$$

Example 2.56. Determine the convergence or divergence of $\sum_{n=1}^{\infty}\left[\frac{4n-5}{2n+1}\right]^n$.

Solution 2.56. The series diverges, since

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left[\frac{4n-5}{2n+1}\right]^n} = \lim_{n\to\infty}\frac{4n-5}{2n+1} = 2 > 1$$

Exercise 2.6.

1. Determine whether the series is convergent or divergent.

(1) $\sum_{n=1}^{\infty}\left[\frac{1}{\ln(1+n)}\right]^n$ (2) $\sum_{n=1}^{\infty}\frac{1}{n!}$ (3) $\sum_{n=1}^{\infty}\frac{(2n)!}{n^5}$

(4) $\sum_{n=1}^{\infty} n\left[\frac{3}{4}\right]^{n-1}$ (5) $\sum_{n=1}^{\infty}\left[2\sqrt[n]{n}+1\right]^n$ (6) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)2^n}$

(7) $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$ (8) $\sum_{n=1}^{\infty}\left[1-\frac{1}{n}\right]^{n^2}$ (9) $\sum_{n=1}^{\infty}\frac{\ln n}{e^n}$

2. For what positive values of α does the series $\sum_{n=1}^{\infty}\frac{\alpha^n}{n^\alpha}$ converge?

3. The terms of a series are defined recursively by the equation $a_{n+1} = \frac{5n+1}{4n+3}a_n$ where $a_1 = 2$. Determine whether $\sum a_n$ converges or diverges?

2.7 Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series, (see the summary in the next page). Skill in choosing and applying the various tests will come only with practice. In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test. Below is a set of guidelines for choosing an appropriate test.

Note 2.2. Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types: geometric, $p-$series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n}\right\rvert > 1$ or $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Exercise 2.7. Determine the convergence or divergence of each series.

(1) $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ (2) $\sum_{n=1}^{\infty} \left[\frac{\pi}{6}\right]^n$ (3) $\sum_{n=1}^{\infty} ne^{-n^2}$ (4) $\sum_{n=1}^{\infty} \left[\frac{n+1}{2n+1}\right]^n$

(5) $\sum_{n=1}^{\infty} \frac{1}{1+3n}$ (6) $\sum_{n=0}^{\infty} \frac{n!}{10^n}$ (7) $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ (8) $\sum_{n=1}^{\infty} \frac{2+\cos n}{n}$

2.8 Power Series

Now that we can test many infinite series of numbers for convergence, we can study sums that look like **infinite polynomials**. We call these sums power series because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

We begin with the formal definition, which specifies the notation and terms used for power series.

Definition 2.8.1. Definition of Power Series
If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is called a **power series centered at** c, where c is a constant.

Radius and Interval of Convergence

A power series in can be viewed as a function of x as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where the domain of f is the set of all x for which the power series converges. Determination of the domain of a power series is the primary concern in this section. Of course, every power series converges at its center because

$$f(c) = \sum_{n=0}^{\infty} a_n (c-c)^n = a_0 + 0 + 0 + \cdots = a_0$$

So, c always lies in the domain of f. The following important theorem states that the domain of a power series can take three basic forms: a single point, an interval centered at c, or the entire real line.

Theorem 2.8.1. ***Convergence of a Power Series***
For a power series centered at c, precisely one of the following is true.

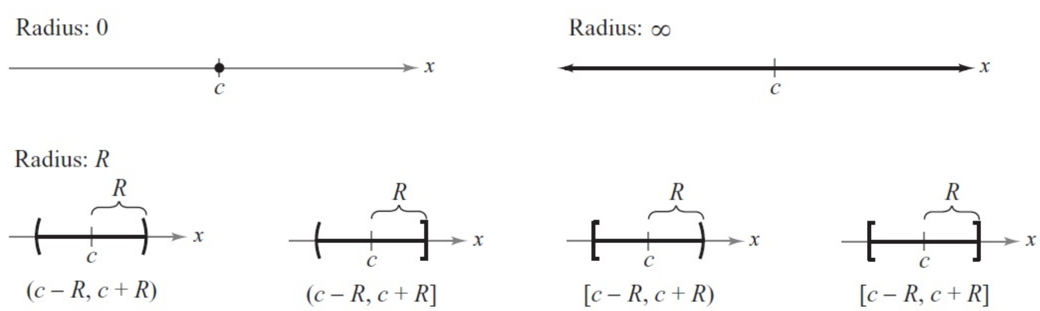
1. *The series converges only at c.*

2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$. The series may or may not converge at either of the endpoints $x = c \pm R$.*

3. *The series converges absolutely for all x.*

The number R is the ***radius of convergence*** *of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the* ***interval of convergence*** *of the power series.*

Note that for a power series whose radius of convergence is a finite number R, Theorem 2.8.1 says nothing about the convergence at the endpoints of the interval of convergence. Each endpoint must be tested separately for convergence or divergence. As a result, the interval of convergence of a power series can take any one of the six forms shown in the figure below.



The usual procedure for finding the radius and interval of convergence of a power series is to apply the Ratio (or Root) Test for absolute convergence. The following examples illustrates how this works.

Example 2.57. Find the interval and radius of convergence of the power series $\sum\limits_{n=0}^{\infty} n!\,x^n$.

Solution 2.57. For $x = 0$, you obtain $f(0) = \sum_{n=0}^{\infty} n!0^n = 1$. For any fixed value of x, let $a_n = n!\,x^n$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = |x|\lim_{n\to\infty}(n+1) = \infty$$

Therefore, by the Ratio Test, the series diverges for $x \neq 0$, and converges only at its center, 0. So, the radius of convergence is $R = 0$.

Example 2.58. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

Solution 2.58. Let $a_n = \frac{x^{2n+1}}{(2n+1)!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\times\frac{(2n+1)!}{x^{2n+1}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)} = 0$$

For any fixed value of x, this limit is 0. So, by the Ratio Test, the series converges for all x. Therefore, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

Example 2.59. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{2^n}$.

Solution 2.59. Letting $a_n = \frac{(x+1)^n}{2^n}$ produces

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\times\frac{2^n}{(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{x+1}{2}\right| = \left|\frac{x+1}{2}\right|$$

By the Ratio Test, the series converges if $\left|\frac{x+1}{2}\right| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints you have

$$\text{when } x = -3 \implies \sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} 1 \qquad \text{Diverges.}$$

$$\text{when } x = 1 \implies \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n \qquad \text{Diverges.}$$

both of which diverge. So, the interval of convergence is $(-3, 1)$

Example 2.60. Find the interval and radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(x-5)^n}{n^2}$.

Solution 2.60. Letting $a_n = \dfrac{(x-5)^n}{n^2}$ produces

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2}\times\frac{n^2}{(x-5)^n}\right| = |x-5|\lim_{n\to\infty}\left[\frac{n}{n+1}\right]^2 = |x-5|$$

By the Ratio Test, the series converges if $|x-5|<1$. So, the radius of convergence is $R=1$. Because the series is centered at $x=5$, it will converge in the interval $(4,6)$. Furthermore, at the endpoints you have

$$\begin{aligned}
&\text{when } x=4 \implies \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \qquad \text{Converges.}\\
&\text{when } x=6 \implies \sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \text{Converges.}
\end{aligned}$$

both of which converge. So, the interval of convergence is $[4,6]$

Example 2.61. Find the interval and radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{n^n}{2^n}x^n$.

Solution 2.61. Letting $a_n = \dfrac{n^n}{2^n}x^n$ produces

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{n^n}{2^n}x^n\right|^{\frac{1}{n}} = |x|\lim_{n\to\infty}\frac{n}{2} = \infty \quad \text{for} \quad x\neq 0$$

So, the series only converges only it $x=0$, and the radius of converges is $R=0$.

Differentiation and Integration of Power Series

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was done in the context of power series, especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoulli all used power series extensively in calculus.

Once you have defined a function with a power series, it is natural to wonder how you can determine the characteristics of the function. Is it continuous? Differentiable? Theorem 2.8.2 answers these questions.

Theorem 2.8.2. ***Properties of Functions Defined by Power Series***
If the function f given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots \end{aligned}$$

has a radius of convergence of $R > 0$, then, on the interval $(c-R, c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots.$
2. $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + \cdots.$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behaviour at the endpoints.

Example 2.62. Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Find the interval of convergence for each of the following.

$$(1) \int f(x)\,dx \qquad (2)\ f(x) \qquad (3)\ f'(x)$$

Solution 2.62. By Theorem 2.8.2, you have

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$$

and

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = C + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} \cdots$$

By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$. Considering the interval $(-1, 1)$, you have the following.

(1) For $\int f(x)\,dx$, the series $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$ converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$.

(2) For $f(x)$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $x = -1$ and diverges for $x = 1$. So, its interval of convergence is $[-1, 1)$.

(3) For $f'(x)$, the series $\sum_{n=1}^{\infty} x^{n-1}$ diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$.

Exercise 2.8.

1. Find the radius and interval of convergence of the series.

   (1) $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ (2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (3) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

   (4) $\sum_{n=1}^{\infty} n!(2x-1)^n$ (5) $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$ (6) $\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$

2. Find all values of $x \in [0, 2\pi]$ for which $\sum_{n=1}^{\infty} \left[\frac{2}{\sqrt{3}}\right]^n \sin^n x$ converges.
   $\left[\textbf{Hint:} \text{ The answer is } \left[0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right]\right]$

3. Give an example of a power series that converges for $x \in [2, 6)$.
   $\left[\textbf{Hint:} \text{ For example, } \sum_{n=1}^{\infty} \frac{(x-4)^n}{n2^n}\right]$

4. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Find the interval of convergence of : f, f', and f''.

2.9 Representation of Functions by Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials.

We start with the geometric power series ($a = 1, r = x$)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for} \quad |x| < 1 \tag{2.9.1}$$

Example 2.63. Find a power series for $f(x) = \dfrac{1}{1+x^2}$, centered at 0.

Solution 2.63. Replacing x by $-x^2$ in Equation 2.9.1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Because this is a geometric series, it converges when $\left|-x^2\right| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the interval of convergence is $(-1, 1)$.

Example 2.64. Find a power series for $f(x) = \dfrac{4}{2+x}$, centered at 0.

Solution 2.64. In order to put this function in the form of the left side of Equation 2.9.1, we first factor a 2 from the denominator:

$$\frac{4}{2+x} = \frac{4}{2\left[1+\frac{x}{2}\right]} = \frac{2}{1-\left[-\frac{x}{2}\right]} = \sum_{n=0}^{\infty} 2\left[-\frac{x}{2}\right]^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n-1}}$$

This power series converges when $\left|-\dfrac{x}{2}\right| < 1$ which implies that the interval of convergence is $(-2, 2)$.

Example 2.65. Find a power series for $f(x) = \dfrac{1}{x}$, centered at 1.

Solution 2.65. In order to put this function in the form of the left side of Equation 2.9.1, we first add and subtract 1 from the denominator:

$$\frac{1}{x}=\frac{1}{1-1+x}=\frac{1}{1-(1-x)}=\sum_{n=0}^{\infty}[-(x-1)]^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

This power series converges when $|x-1|<1$ which implies that the interval of convergence is $(0,2)$.

Example 2.66. Find a power series for $f(x)=\dfrac{4x^3}{2+x}$, centered at 0.

Solution 2.66. Since this function is just x^3 times the function in Example 2.64, all we have to do is to multiply that series by x^3:

$$\frac{4x^3}{2+x}=x^3\times\frac{4}{2+x}=x^3\times\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n-1}}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+3}}{2^{n-1}}$$

As in Example 2.64, the interval of convergence is $(-2,2)$.

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. For simplicity, the following properties are stated for a series centered at 0.

Theorem 2.9.1. ***Operations with Power Series***
Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $g(x)=\sum_{n=0}^{\infty}b_nx^n$.

1. $f(mx)=\sum_{n=0}^{\infty}a_nm^nx^n$.
2. $f\left(x^m\right)=\sum_{n=0}^{\infty}a_nx^{mn}$.
3. $f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n\pm b_n)x^n$.

The operations described above can change the interval of convergence for the resulting series. For example, in the following addition, the interval of convergence for the sum is the intersection of the intervals of convergence of the two original series.

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{(-1,1)} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{(-2,2)} = \underbrace{\sum_{n=0}^{\infty} \left(1+\frac{1}{2^n}\right) x^n}_{(-1,1)\cap(-2,2)=(-1,1)}$$

Example 2.67. Find a power series, centered at 0, for $f(x) = \dfrac{3x-1}{x^2-1}$.

Solution 2.67. Using partial fractions, you can write as $f(x) = \dfrac{2}{x+1} + \dfrac{1}{x-1}$. By adding the two geometric power series

$$\frac{2}{x+1} = \frac{2}{1-(-x)} = 2\sum_{n=0}^{\infty} (-1)^n x^n \quad \text{where} \quad |x| < 1$$

$$\text{and} \quad \frac{1}{x-1} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n \quad \text{where} \quad |x| < 1$$

you obtain the power series $\dfrac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} \left[2(-1)^n - 1\right] x^n$, where the interval of convergence for this power series is $(-1,1)$.

Example 2.68. Find a power series for $f(x) = \ln x$, centered at 1.

Solution 2.68. Since $\displaystyle\int \frac{1}{x}\,dx = \ln x + C$, and from Example 2.65, you know that $\dfrac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$, then

$$\begin{aligned} \ln x &= C + \int \frac{1}{x}\,dx = C + \int \left[\sum_{n=0}^{\infty} (-1)^n (x-1)^n\right] dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \end{aligned}$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Example 2.69. Find a power series for $g(x) = \tan^{-1} x$, centered at 0.

Solution 2.69. Because $\dfrac{d}{dx}\left[\tan^{-1} x\right] = \dfrac{1}{1+x^2}$, and from Example 2.63 you know that $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, you obtain

$$\begin{aligned}\tan^{-1} x &= C + \int \frac{1}{1+x^2}\,dx = C + \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

By letting $x=0$, you can conclude that $C=0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Exercise 2.9.

1. Find a power series for the function, centered at c, and determine the interval of convergence.

   a) $f(x) = \dfrac{1}{(1-x)^2}$ at $c=0$. $\left[\textbf{Hint:}\ \dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)\right]$.

   b) $f(x) = \ln(1+x)$ at $c=0$. $\left[\textbf{Hint:}\ \ln(1+x) = \displaystyle\int \frac{1}{1+x}\,dx\right]$.

   c) $f(x) = \dfrac{1}{3-x}$ at $c=1$. $\left[\textbf{Hint:}\ \dfrac{1}{3-x} = \dfrac{1/2}{1-\left(\frac{x-1}{2}\right)}\right]$.

   d) $f(x) = \dfrac{4x}{x^2+2x-3}$ at $c=0$. $\left[\textbf{Hint:}\ \dfrac{4x}{x^2+2x-3} = \dfrac{1}{x-1} + \dfrac{3}{x+3}\right]$.

   e) $f(x) = \dfrac{2}{(1+x)^3}$ at $c=0$. $\left[\textbf{Hint:}\ \dfrac{2}{(1+x)^3} = \dfrac{d^2}{dx^2}\left(\dfrac{1}{1+x}\right)\right]$.

   f) $f(x) = \ln\left(1-x^2\right)$ at $c=0$. $\left[\textbf{Hint:}\ \ln\left(1-x^2\right) = \ln(1-x) + \ln(1+x)\right]$.

   g) $f(x) = \dfrac{x}{(1+4x)^2}$ at $c=0$.

   h) $f(x) = \left(\dfrac{x}{2-x}\right)^3$ at $c=0$.

2. Suppose that the series $\sum a_n x^n$ has radius of convergence 2, and the series $\sum b_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (a_n + b_n) x^n$?

3. Suppose that the radius of convergence of the power series $\sum a_n x^n$ is R. What is the radius of convergence of the power series $\sum a_n x^{2n}$?

4. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. $\left[\textbf{Hint:} \text{ Show that } \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}\right]$.

2.10 Taylor and Maclaurin Series

We saw in the previous section that functions such as $f(x) = \tan^{-1} x$ can be represented as power series. These power series give us a certain tangible insight into the function represented and they allow us to approximate the values of $f(x)$ to any desired degree of accuracy. Thus, it is desirable to develop general methods for finding power series representations.

Theorem 2.10.1. ***The Form of a Convergent Power Series***
If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for all x in an open interval I containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Definition 2.10.1. Definition of Taylor and Maclaurin Series
If a function f has derivatives of all orders at $x = c$, then the series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the Taylor series for $f(x)$ at c. Moreover, if $c = 0$, then the series is the Maclaurin series for f.

If you know the pattern for the coefficients of the Taylor polynomials for a function, you can extend the pattern easily to form the corresponding Taylor series.

Example 2.70. Find the Maclaurin series for $f(x) = e^x$.

Solution 2.70. The nth derivative $f(x)$ is $f^{(n)}(x) = e^x$ for all n and thus

$$f(0) = f'(0) = f''(0) = \cdots = e^0 = 1.$$

Therefore, the coefficients of the Maclaurin series are $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$ and the Maclaurin series is $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Example 2.71. Find the Maclaurin series for $f(x) = \sin x$.

Solution 2.71. For $f(x) = \sin x$, we have

$$f^{(2n)}(x) = (-1)^n \sin x \quad \text{and} \quad f^{(2n+1)}(x) = (-1)^n \cos x$$

Therefore, $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$. Hence, the coefficients of the Maclaurin series are $a_{2n+1} = \dfrac{(-1)^n}{(2n+1)!}$ and the Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Since a Taylor series is a power series, we may differentiate and integrate a Taylor series term by term within its interval of convergence. We may also multiply two Taylor series or substitute one Taylor series into another. This leads to shortcuts for generating new Taylor series from known ones. The following list provides the Maclaurin series for several elementary functions with the corresponding intervals of convergence.

Function $f(x)$	**Maclaurin Series**	**Converges to** $f(x)$ **for**
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$\lvert x\rvert < 1$
$\dfrac{1}{1+x}$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$	$\lvert x\rvert < 1$
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\lvert x\rvert < 1$ and $x = 1$
$\tan^{-1} x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \le 1$

Example 2.72. Find the Maclaurin series for $f(x) = x^2 e^x$.

Solution 2.72. We obtain the Maclaurin series of $f(x)$ by multiplying the known Maclaurin series for e^x by x^2:

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}$$

Example 2.73. Find the Maclaurin series for $f(x) = \cos\sqrt{x}$.

Solution 2.73. Using the power series $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ you can replace x by $\sqrt{x}$ to obtain the series $\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$. This series converges for $x \geq 0$.

Example 2.74. Find the Maclaurin series for $f(x) = \sin^2 x$.

Solution 2.74. Write $\sin^2 x$ as

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and then, use the Maclaurin series for $\cos x$ as follows.

$$\begin{aligned} \sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} \\ &= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n-1} x^{2n} \end{aligned}$$

Example 2.75. Find Taylor series for $f(x) = \ln x$ at $c = 1$.

Solution 2.75. We begin by letting $t = x - 1$ that transforms f to $\ln(t+1)$, and now looking for the Maclaurin series at $c = 0$. So,

$$f(x) = \ln x = \ln(t+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

Example 2.76. Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

Solution 2.76. the Maclaurin series for e^{-x^2} is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!}$. Now we integrate to obtain

$$\int e^{-x^2}\,dx = C + \int \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!}\right] dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)n!}$$

Example 2.77. If $f(x) = \sin\left(x^3\right)$, find $f^{(15)}(0)$.

Solution 2.77. It would be far too much to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. Since

$$f(x) = \sin\left(x^3\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

then the coefficient of x^{15} is $\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}$, and hence

$$f^{(15)}(0) = \frac{15!}{5!} = 10\,897\,286\,400$$

Example 2.78. Find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}$.

Solution 2.78. Remember that $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. So,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{3}{5}\right)^n}{n} = \ln\left(1+\frac{3}{5}\right) = \ln\left(\frac{8}{5}\right)$$

Exercise 2.10.

1. Find Maclaurin series for the given function.

   (1) $\sin x \cos x$ (2) $\frac{1}{1+5x}$ (3) $\cos^2 x$

   (4) $\ln\left[\frac{1+x}{1-x}\right]$ (5) $x^2 e^{-3x}$

2. Find Taylor series for the given function at the indicated value of c.

   (1) $\dfrac{1}{1+x}$, $(c=4)$ (2) $\sin x$, $(c=\pi/2)$ (3) $\ln(1+x)$, $(c=2)$

   (4) e^x, $(c=1)$ (5) $\sqrt{x}$, $(c=1)$

3. Find $f^{(10)}(0)$ for $f(x)=x^4\sin\left(x^2\right)$.

4. Does $f(x)=\cot x$ possess a Maclaurin series representation ?

5. Use Maclaurin series to evaluate $\displaystyle\lim_{x\to0}\frac{1+x-e^x}{1-\cos x}$.

6. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{36^n(2n)!}$

7. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}n^3x^n$ and find its sum.

8. Find the sum of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots$$

   where the terms are reciprocals of the positive integers whose only prime factors are 2’s and 3’s.

APPENDIX A

Indeterminate Forms and L'Hôspital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as l'Hôspital's Rule, after Guillaume de l'Hôspital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

A.1 Indeterminate Form $\mathbf{0/0}, \infty/\infty$

If the functions $f(x)$ and $g(x)$ are both zero or both $\pm\infty$ at $x = a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$ or ∞/∞, a meaningless expressions (**indeterminate forms**), that we cannot evaluate. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. L'Hôspital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

Theorem A.1.1. *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or $\pm\infty$.

L'Hôspital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of and before using L'Hôspital's Rule.

L'Hôspital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

Example A.1. Find

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

Solution A.1. Since $\lim_{x \to 1} \ln x = 0$ and $\lim_{x \to 1}(x-1) = 0$, then we can apply L'Hôspital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Example A.2. Calculate

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

Solution A.2. We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, so L'Hôspital's Rule gives:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}\left(x^2\right)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of L'Hôspital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

Example A.3. Calculate

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$$

Solution A.3. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} \sqrt[3]{x} = \infty$ as $x \to \infty$, L'Hôspital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying L'Hôspital's Rule a second time as we did in the previous example, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

Example A.4. Find

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

Solution A.4. Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use L'Hôspital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply L'Hôspital's Rule again:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \\
&= \frac{2}{6} \times \lim_{x\to 0} \sec^2 x \times \lim_{x\to 0} \frac{\tan x}{x} \\
&= \frac{1}{3} \times 1 \times 1 = \frac{1}{1}
\end{aligned}$$

A.2 Indeterminate Products $\mathbf{0} \cdot \pm\infty$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it is not clear what the value of $\lim_{x\to a} \left[f(x)g(x)\right]$, if any, will be. There is a struggle between f and g. If f

wins, the answer will be 0; if g wins, the answer will be $\pm\infty$. Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type** $0\cdot\infty$. We can deal with it by writing the product as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use L'Hôspital's Rule.

Example A.5. Evaluate

$$\lim_{x\to 0^+} x \ln x$$

Solution A.5. The given limit is indeterminate because, as $x \to 0^+$, the first factor x approaches 0 while the second factor $\ln x$ approaches $-\infty$. Writing x as $\frac{1}{1/x}$ we have $1/x \to \infty$ as $x \to 0^+$, so L'Hôspital's Rule gives:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

A.3 Indeterminate Differences $\infty - \infty$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} \left[f(x) - g(x) \right]$$

is called an **indeterminate form of type** $\infty - \infty$. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be ∞ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

Example A.6. Evaluate

$$\lim_{x\to (\pi/2)^-} (\sec x - \tan x)$$

Solution A.6. First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\begin{aligned}
\lim_{x\to(\pi/2)^-} (\sec x - \tan x) &= \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\
&= \lim_{x\to(\pi/2)^-} \frac{1-\sin x}{\cos x} \\
&= \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0
\end{aligned}$$

A.4 Indeterminate Powers $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$

These several indeterminate forms arise from the limit

$$\lim_{x\to a} \left[f(x)\right]^{g(x)}$$

Each of these three cases can be treated by writing the function as an exponential:

$$\left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$$

and then

$$\lim_{x\to a} \left[f(x)\right]^{g(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}$$

where the indeterminate product $g(x)\ln f(x)$ is of type $0\cdot\infty$.

Example A.7. Calculate

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$$

Solution A.7. First notice that as $x \to 0^+$, we have $1+\sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$(1+\sin 4x)^{\cot x} = e^{\cot x \ln(1+\sin 4x)}$$

Then

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = e^{\lim_{x\to 0^+} \cot x \ln(1+\sin 4x)}$$

Since

$$\begin{aligned}
\lim_{x\to 0^+} \cot x \ln(1+\sin 4x) &= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \\
&= \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4
\end{aligned}$$

then

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = e^4$$

Example A.8. Find

$$\lim_{x \to 0^+} x^x$$

Solution A.8. Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed by writing the function as an exponential

$$x^x = e^{x \ln x}$$

and then

$$\lim_{x \to 0^+} x^x = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

ALGEBRA

Lines

Slope of the line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Point-slope equation of line through $P_1 = (x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Point-point equation of line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$y - y_1 = m(x - x_1) \quad \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Lines of slope m_1 and m_2 are parallel if and only if $m_1 = m_2$.
Lines of slope m_1 and m_2 are perpendicular if and only if $m_1 = -\frac{1}{m_2}$.

Circles

Equation of the circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance and Midpoint Formulas

Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Laws of Exponents

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n} \qquad (x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n} \qquad (xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Special Factorizations

$$x^2 - y^2 = (x + y)(x - y)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

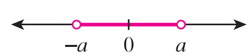
where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot k}$

Quadratic Formula

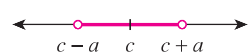
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.
If $a < b$, then $a + c < b + c$.
If $a < b$ and $c > 0$, then $ca < cb$.
If $a < b$ and $c < 0$, then $ca > cb$.
$|x| = x \quad \text{if } x \geq 0$
$|x| = -x \quad \text{if } x \leq 0$



$|x| < a$ means $-a < x < a$.

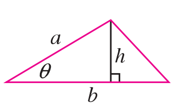


$|x - c| < a$ means $c - a < x < c + a$.

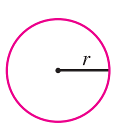
GEOMETRY

Formulas for area A, circumference C, and volume V

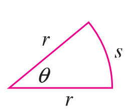
Triangle
$A = \frac{1}{2}bh$
$= \frac{1}{2}ab\sin\theta$



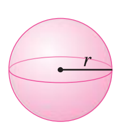
Circle
$A = \pi r^2$
$C = 2\pi r$



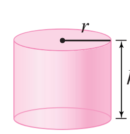
Sector of Circle
$A = \frac{1}{2}r^2\theta$
$s = r\theta$
(θ in radians)



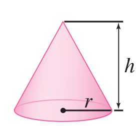
Sphere
$V = \frac{4}{3}\pi r^3$
$A = 4\pi r^2$



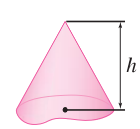
Cylinder
$V = \pi r^2 h$



Cone
$V = \frac{1}{3}\pi r^2 h$
$A = \pi r\sqrt{r^2 + h^2}$



Cone with arbitrary base
$V = \frac{1}{3}Ah$
where A is the area of the base



Pythagorean Theorem: For a right triangle with hypotenuse of length c and legs of lengths a and b, $c^2 = a^2 + b^2$.

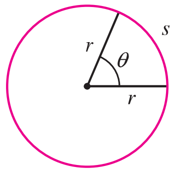
TRIGONOMETRY

Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad 1$ rad $= \dfrac{180°}{\pi}$

$s = r\theta \quad (\theta$ in radians)

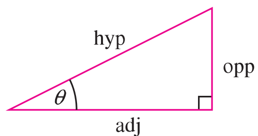


Right Triangle Definitions

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\text{adj}}{\text{opp}}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hyp}}{\text{opp}}$



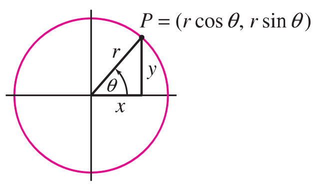
Trigonometric Functions

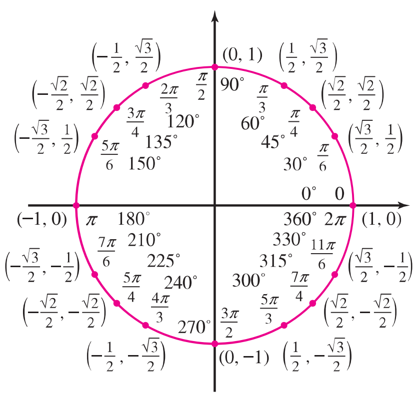
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

$\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$





Fundamental Identities

$\sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta$

$1 + \tan^2\theta = \sec^2\theta \qquad \cos(-\theta) = \cos\theta$

$1 + \cot^2\theta = \csc^2\theta \qquad \tan(-\theta) = -\tan\theta$

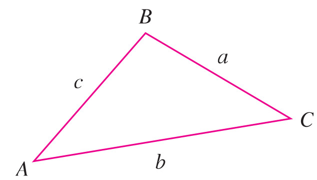
$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$

$\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta \qquad \tan(\theta + \pi) = \tan\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

Double-Angle Formulas

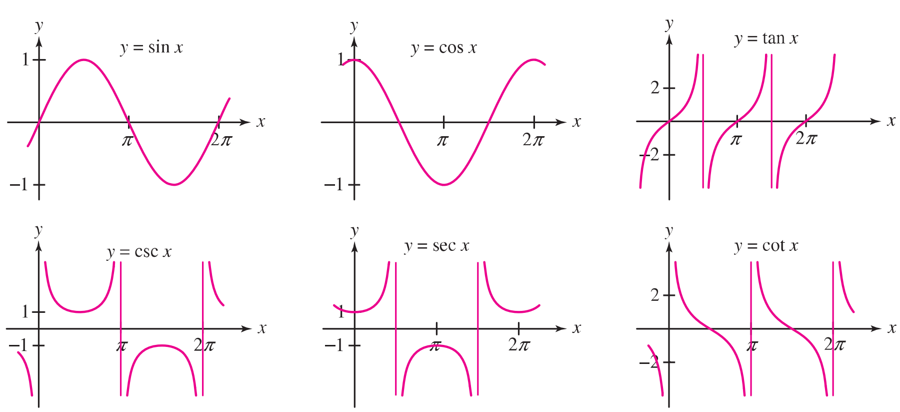
$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

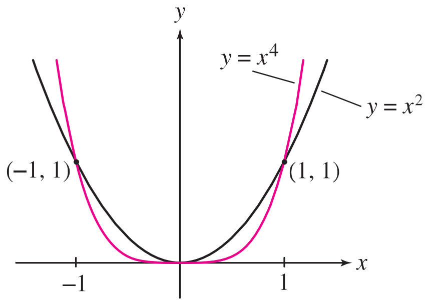
Graphs of Trigonometric Functions



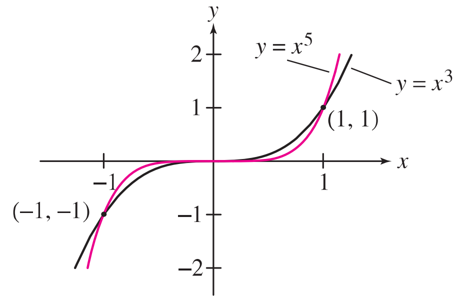
ELEMENTARY FUNCTIONS

Power Functions $f(x) = x^a$

$f(x) = x^n$, n a positive integer

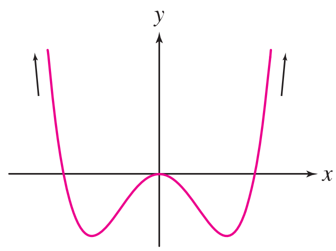


n even



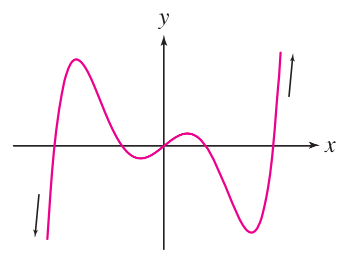
n odd

Asymptotic behavior of an even polynomial function



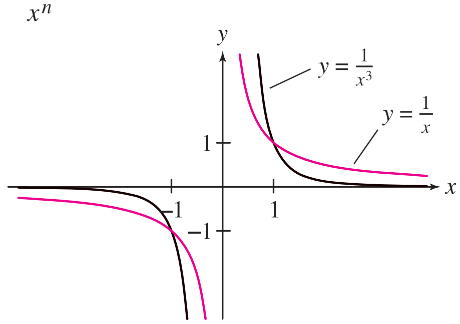
n even

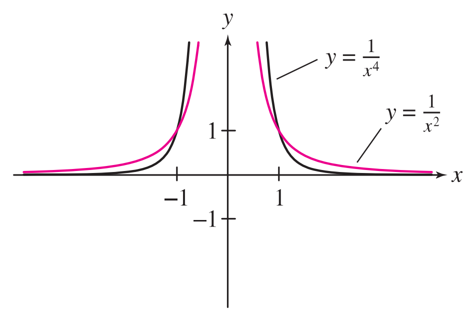
Asymptotic behavior of an odd polynomial function



n odd

$f(x) = x^{-n} = \dfrac{1}{x^n}$





Inverse Trigonometric Functions

$\arcsin x = \sin^{-1} x = \theta$

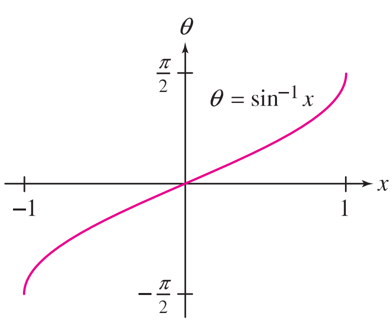
$\Leftrightarrow \quad \sin\theta = x, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

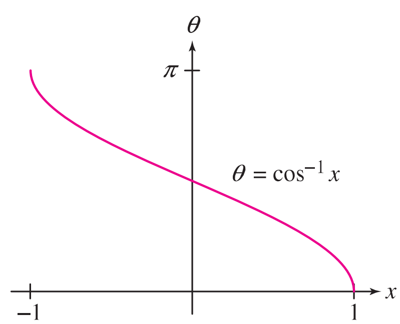
$\arccos x = \cos^{-1} x = \theta$

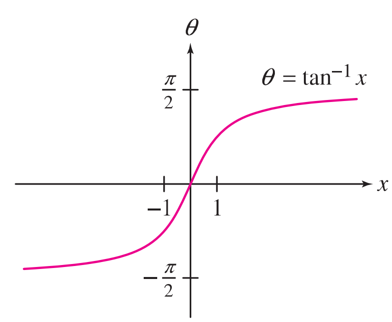
$\Leftrightarrow \quad \cos\theta = x, \quad 0 \le \theta \le \pi$

$\arctan x = \tan^{-1} x = \theta$

$\Leftrightarrow \quad \tan\theta = x, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$







Exponential and Logarithmic Functions

$\boxed{\log_a x = y \quad \Leftrightarrow \quad a^y = x}$

$\log_a(a^x) = x \qquad a^{\log_a x} = x$

$\log_a 1 = 0 \qquad \log_a a = 1$

$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$

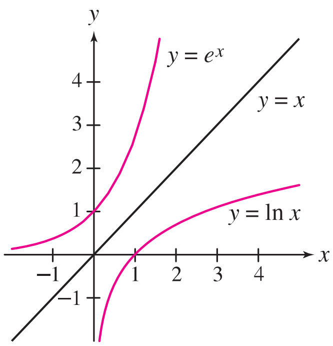
$\ln(e^x) = x \qquad e^{\ln x} = x$

$\ln 1 = 0 \qquad \ln e = 1$

$\log_a(xy) = \log_a x + \log_a y$

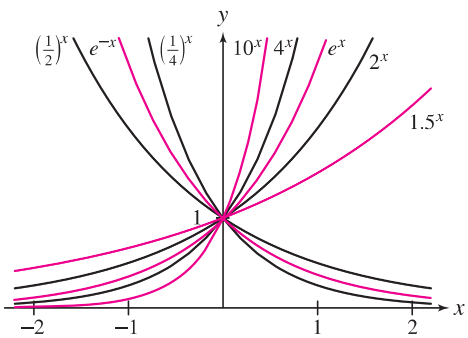
$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^r) = r\log_a x$



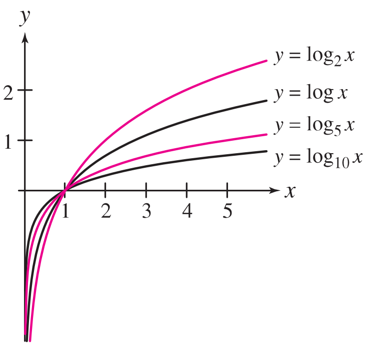
$$\lim_{x\to\infty} a^x = \infty, \quad a > 1$$

$$\lim_{x\to\infty} a^x = 0, \quad 0 < a < 1$$



$$\lim_{x\to-\infty} a^x = 0, \quad a > 1$$

$$\lim_{x\to-\infty} a^x = \infty, \quad 0 < a < 1$$



$$\lim_{x\to 0^+} \log_a x = -\infty$$

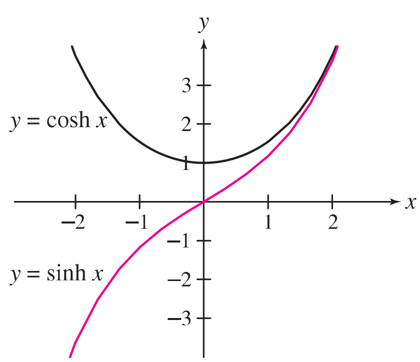
$$\lim_{x\to\infty} \log_a x = \infty$$

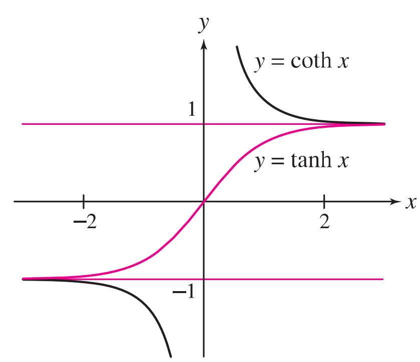
Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$





$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Inverse Hyperbolic Functions

$y = \sinh^{-1} x \quad \Leftrightarrow \quad \sinh y = x$

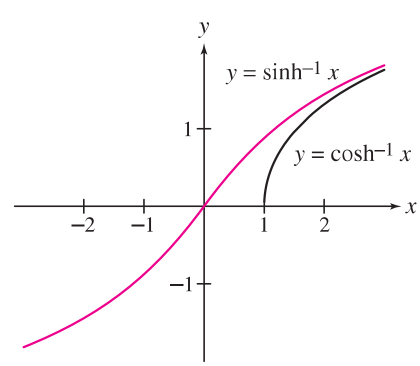
$y = \cosh^{-1} x \quad \Leftrightarrow \quad \cosh y = x \ \text{ and } \ y \ge 0$

$y = \tanh^{-1} x \quad \Leftrightarrow \quad \tanh y = x$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x > 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$



DIFFERENTIATION

Differentiation Rules

1. $\dfrac{d}{dx}(c) = 0$

2. $\dfrac{d}{dx}x = 1$

3. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

4. $\dfrac{d}{dx}[cf(x)] = cf'(x)$

5. $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

6. $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

7. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

8. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

9. $\dfrac{d}{dx}f(x)^n = nf(x)^{n-1}f'(x)$ (General Power Rule)

10. $\dfrac{d}{dx}f(kx+b) = kf'(kx+b)$

11. $g'(x) = \dfrac{1}{f'(g(x))}$ where $g(x)$ is the inverse $f^{-1}(x)$

12. $\dfrac{d}{dx}\ln f(x) = \dfrac{f'(x)}{f(x)}$

Trigonometric Functions

13. $\dfrac{d}{dx}\sin x = \cos x$

14. $\dfrac{d}{dx}\cos x = -\sin x$

15. $\dfrac{d}{dx}\tan x = \sec^2 x$

16. $\dfrac{d}{dx}\csc x = -\csc x \cot x$

17. $\dfrac{d}{dx}\sec x = \sec x \tan x$

18. $\dfrac{d}{dx}\cot x = -\csc^2 x$

Inverse Trigonometric Functions

19. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

20. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

21. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

22. $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{x\sqrt{x^2-1}}$

23. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

24. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

Exponential and Logarithmic Functions

25. $\dfrac{d}{dx}(e^x) = e^x$

26. $\dfrac{d}{dx}(a^x) = (\ln a)a^x$

27. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$

28. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{(\ln a)x}$

Hyperbolic Functions

29. $\dfrac{d}{dx}(\sinh x) = \cosh x$

30. $\dfrac{d}{dx}(\cosh x) = \sinh x$

31. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

32. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

33. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

34. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Inverse Hyperbolic Functions

35. $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

36. $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$

37. $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$

38. $\dfrac{d}{dx}(\operatorname{csch}^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2+1}}$

39. $\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = -\dfrac{1}{x\sqrt{1-x^2}}$

40. $\dfrac{d}{dx}(\coth^{-1} x) = \dfrac{1}{1-x^2}$

INTEGRATION

Substitution

If an integrand has the form $f(u(x))u'(x)$, then rewrite the entire integral in terms of u and its differential $du = u'(x)\,dx$:

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du$$

Integration by Parts Formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

TABLE OF INTEGRALS

Basic Forms

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$

5. $\displaystyle\int \sin u\,du = -\cos u + C$

6. $\displaystyle\int \cos u\,du = \sin u + C$

7. $\displaystyle\int \sec^2 u\,du = \tan u + C$

8. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

9. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

10. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

11. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$

15. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

16. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

Exponential and Logarithmic Forms

17. $\displaystyle\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

18. $\displaystyle\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

19. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

20. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

21. $\displaystyle\int \ln u\,du = u\ln u - u + C$

22. $\displaystyle\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

23. $\displaystyle\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

24. $\displaystyle\int \sinh u\,du = \cosh u + C$

25. $\displaystyle\int \cosh u\,du = \sinh u + C$

26. $\displaystyle\int \tanh u\,du = \ln\cosh u + C$

27. $\displaystyle\int \coth u\,du = \ln|\sinh u| + C$

28. $\displaystyle\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

29. $\displaystyle\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$

30. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

31. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

32. $\displaystyle\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

33. $\displaystyle\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Trigonometric Forms

34. $\displaystyle\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

35. $\displaystyle\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

36. $\displaystyle\int \tan^2 u\,du = \tan u - u + C$

37. $\displaystyle\int \cot^2 u\,du = -\cot u - u + C$

38. $\displaystyle\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

39. $\displaystyle\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

40. $\displaystyle\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

41. $\displaystyle\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

42. $\displaystyle\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

43. $\displaystyle\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

44. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$

45. $\displaystyle\int \cos^n u\,du = \frac{1}{2}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$

46. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$

47. $\displaystyle\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$

48. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$

49. $\displaystyle\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$

50. $\displaystyle\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

51. $\displaystyle\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

52. $\displaystyle\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

53. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

54. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

55. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

56. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

57. $\displaystyle\int \sin^n u\cos^m u\,du$

$$= -\frac{\sin^{n-1}u\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\cos^m u\,du$$

$$= \frac{\sin^{n+1}u\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2}u\,du$$

Inverse Trigonometric Forms

58. $\displaystyle\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

59. $\displaystyle\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

60. $\displaystyle\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C$

61. $\displaystyle\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

62. $\displaystyle\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

63. $\displaystyle\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

64. $\displaystyle\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

65. $\displaystyle\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\displaystyle\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Forms Involving $\sqrt{a^2-u^2}$, $a > 0$

67. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

68. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

69. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

70. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$

71. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

72. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

73. $\displaystyle\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

74. $\displaystyle\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

75. $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving $\sqrt{u^2-a^2}$, $a > 0$

76. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

77. $\displaystyle\int u^2\sqrt{u^2-a^2}\,du$

$$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

78. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

79. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

80. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\displaystyle\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\displaystyle\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

83. $\displaystyle\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Forms Involving $\sqrt{a^2+u^2}$, $a > 0$

84. $\displaystyle\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

85. $\displaystyle\int u^2\sqrt{a^2+u^2}\,du$

$$= \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$$

86. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

87. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

88. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

89. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

90. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

91. $\displaystyle\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

92. $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Forms Involving $a + bu$

93. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu - a\ln|a+bu|\right) + C$

94. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

95. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

96. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

97. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

98. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

99. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

100. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

101. $\displaystyle\int u^n\sqrt{a+bu}\,du$

$$= \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

102. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

103. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

104. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C,\quad \text{if } a>0$

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C,\quad \text{if } a<0$$

105. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

106. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

107. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

108. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

109. $\displaystyle\int u\sqrt{2au-u^2}\,du$

$$= \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

110. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

111. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

ESSENTIAL THEOREMS

Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in [a, b]$ such that $f(c) = M$.

Mean Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there exists at least one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Extreme Values on a Closed Interval

If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains both a minimum and a maximum value on $[a, b]$. Furthermore, if $c \in [a, b]$ and $f(c)$ is an extreme value (min or max), then c is either a critical point or one of the endpoints a or b.

The Fundamental Theorem of Calculus, Part I

Assume that $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Fundamental Theorem of Calculus, Part II

Assume that $f(x)$ is a continuous function on $[a, b]$. Then the area function $A(x) = \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, that is,

$$A'(x) = f(x) \quad \text{or equivalently} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.