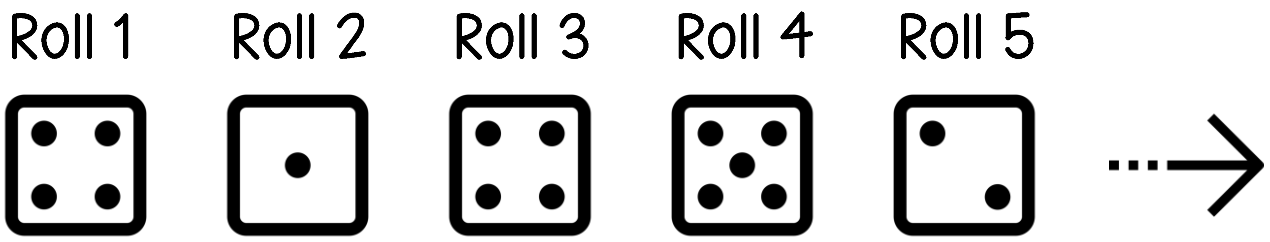
Course: Applied Probability

Chapter: [1]
Markov Chains

Section: [1.1]
What Is a Stochastic Process?

Stochastic Process

Interest: Sometimes we are interested in how a random variable changes **over time**.



Definition: A **stochastic (random) process** involves a sequence of experiments where the outcome of each experiment is not certain.

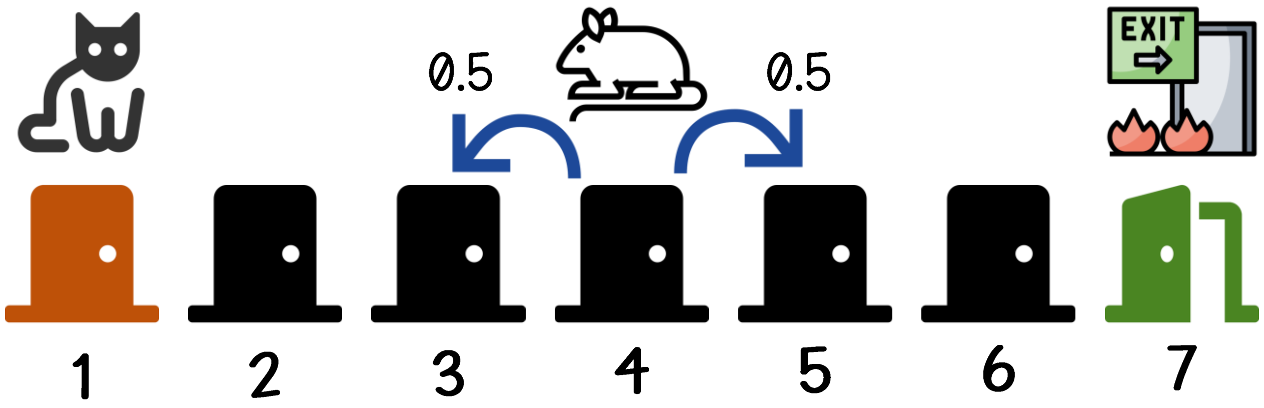
Stochastic Process

Example: (Cat & Mouse)

There are seven doors arranged in a straight line. A mouse initiates the game at the central door, which is door 4. Door 1 houses a cat, and the ultimate goal is to reach door 7 for freedom. Each day, the mouse makes a random decision to move either one door to the left or one door to the right, with an equal 50% chance for each direction. The game continues until one of two conditions is met: if the mouse encounters the cat at door 1, the game ends with the cat catching the mouse; conversely, if the mouse successfully reaches door 7, it secures its freedom, concluding the game.

Stochastic Process

Example: (Cat & Mouse)



* The door where the mouse resides is referred to as the system's **current state**.

* Let X_t be the mouse position after t days.

* The state space of $\{X_t : t \in T\}$ is:

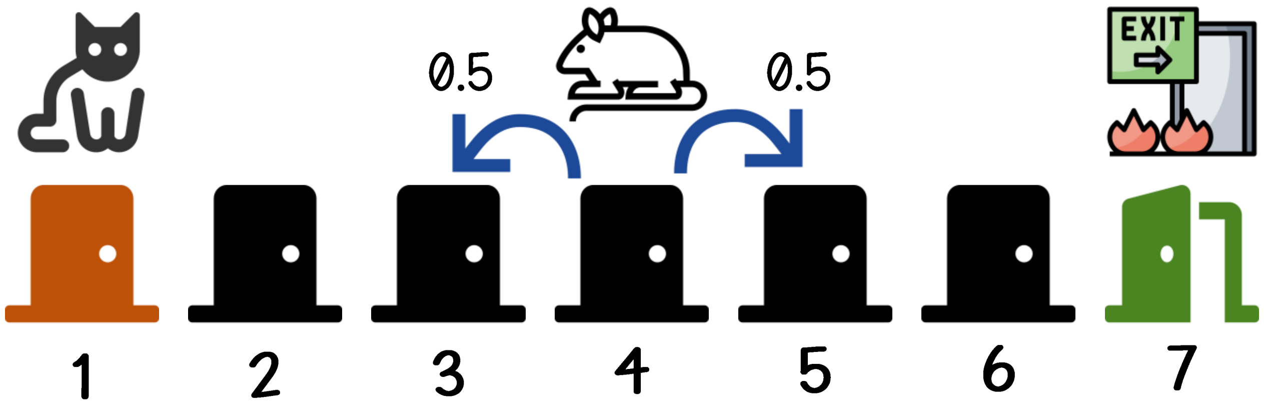
$$S = \{1,2,3,4,5,6,7\}$$

$$T = \{0,1,2,3,\cdots\}$$

Day	Door
0	4
1	5
2	4
3	3
4	2
5	3
6	2
7	1

Stochastic Process

Example: (Cat & Mouse)



* Note that X_t is a **discrete random variable.**

* In this example, being in a state at time $t+1$ depends on the state at time t and does not depend on the states the chain passed through on the way to it at time t.

Day	Door
0	4
1	5
2	4
3	3
4	2
5	3
6	2
7	1

Stochastic Process

Example: Let X_t be the number of customers in a supermarket at any period of time t.

* Note that X_t is a **continuous random variable**.

* The state space of $\{X_t : t \in T\}$ is:

$S = \{0,1,2,3,\cdots\}$ $\quad$ $T = [0,\infty)$

Time	Customers
12:15 PM	8
5 minutes after opening	3
From 10:10 To 10:22	4

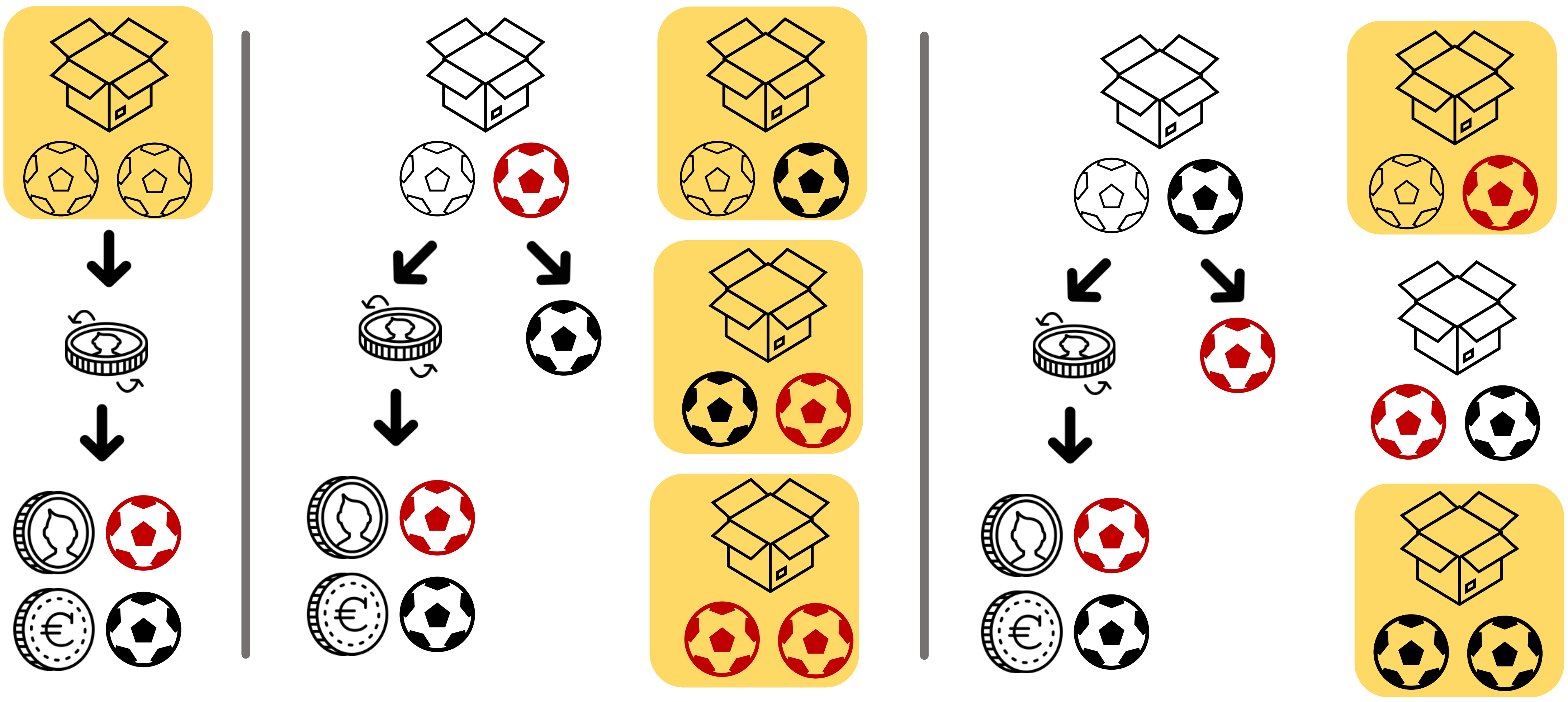
Stochastic Process

Example: (Balls from Urn)

An urn contains two unpainted balls at present. We choose a ball at random and flip a coin. If the chosen ball is unpainted and the coin comes up heads, we paint the chosen unpainted ball red; if the chosen ball is unpainted and the coin comes up tails, we paint the chosen unpainted ball black. If the ball has already been painted, then (whether heads or tails has been tossed) we change the color of the ball (from red to black or from black to red). To model this situation as a stochastic process, we define time t to be the time after the coin has been flipped for the t^{th} time and the chosen ball has been painted.

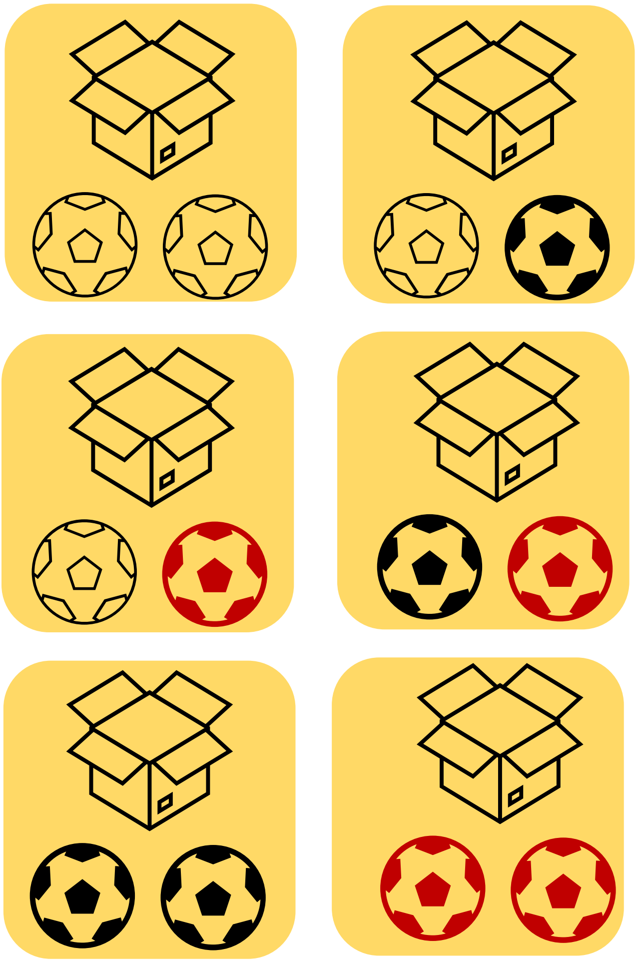
Stochastic Process

Example: (Balls from Urn)



Stochastic Process

Example: (Balls from Urn)



The state at any time may be described by the 3-digits number

u r b

number of unpainted balls

number of red balls

number of black balls

$$X_0 = 200$$

After the first coin toss

$X_1 = 110$ or $X_1 = 101$

If $X_t = 020$ then $X_{t+1} = 011$

Course: Applied Probability

Chapter: [1]

Markov Chains

Section: [1.2]

What Is a Markov Chain?

Markov Chain

Concept: A **Markov Chain** is a stochastic process used to describe a sequence of events where the outcome of each event **depends only on the previous one**, exhibiting a **memoryless property**.

Definition: A discrete—time stochastic process is a **Markov chain** if, for $t = 0,1,2,\cdots$ and all states,

$$P(X_{t+1} = i_{t+1} | X_t = i_t, X_{t-1} = i_{t-1}, \cdots, X_1 = i_1, X_0 = i_0) \\ = P(X_{t+1} = i_{t+1} | X_t = i_t)$$

Markov Chain

Example: Assume there's a restaurant that serves only three types of foods:

Burger Pizza Sausage

But the restaurant follows a weird rule in serving these foods. On any given day they serve only one of these three items and it depends on what they had served yesterday, and can't serve Pizza in two consecutive days.

Day 1 Day 2 Day 3 Day 4 Day 5

Predict ?

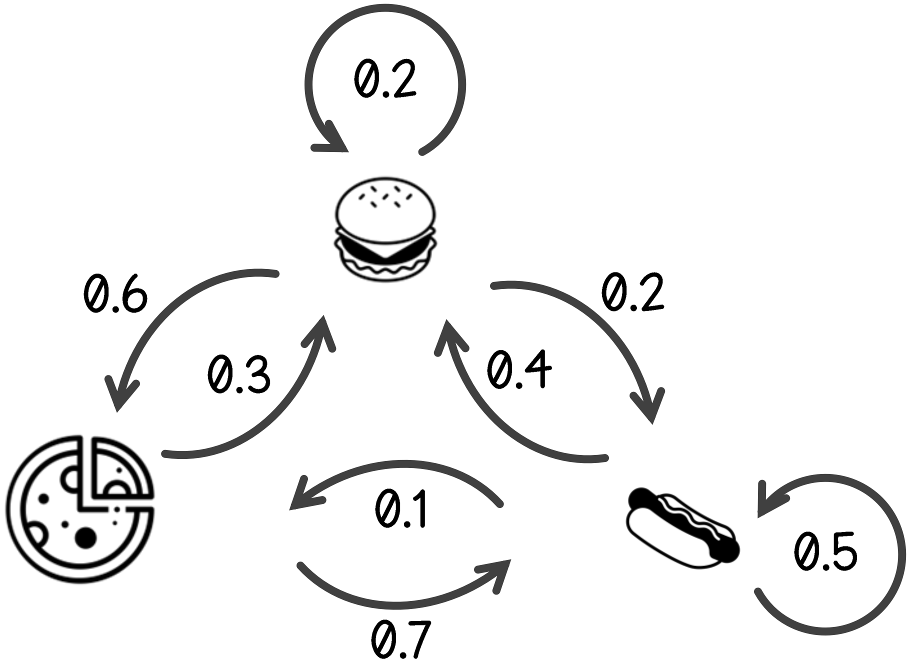
Markov Chain

Example: The states of the system is the set $S = \{Burger, Pizza, Sausage\}$

Each weighted directed arrow is called a **transition** from one state to the other with the given **probability**.

Each number p_{ij} in the diagram is the **probability** that given the system is in state i at time t, it will be in a state j at time t+1.

For all states i and j and all t, $P(X_{t+1} = j | X_t = i)$ is **independent** of the time t.



$$P(X_{t+1} = j | X_t = i) = p_{ij}$$

Markov Chain

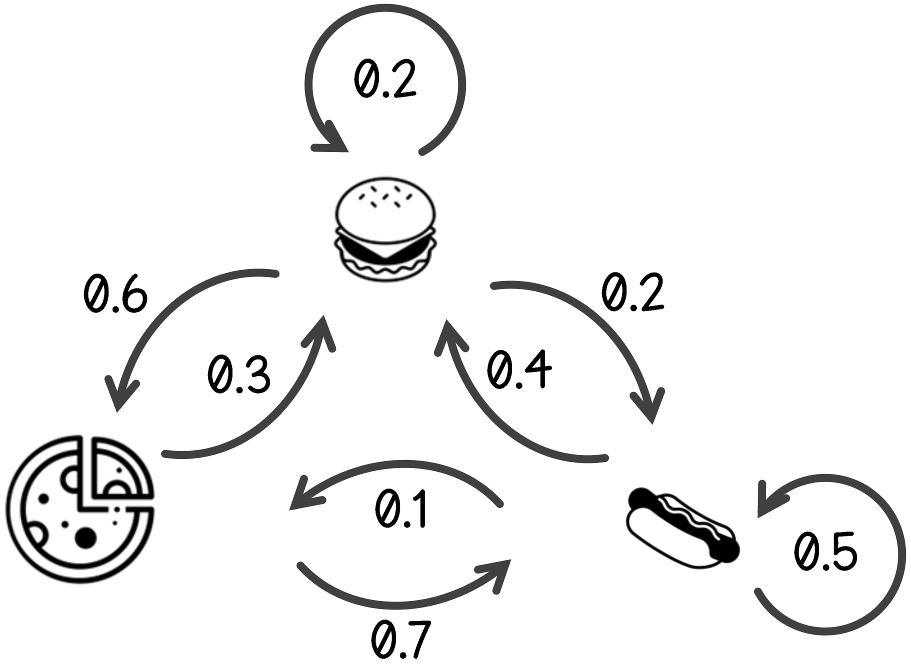
Example: The states of the system is the set
S = {B, Z, S}

Same Order

Next State

Current State

	B	Z	S	
B	0.2	0.6	0.2	$\Sigma = 1$
Z	0.3	0.0	0.7	$\Sigma = 1$
S	0.4	0.1	0.5	$\Sigma = 1$



The probabilities shown in the directed graph can be represented by a **square matrix** called the **transition probabilities** of the Markov chain.

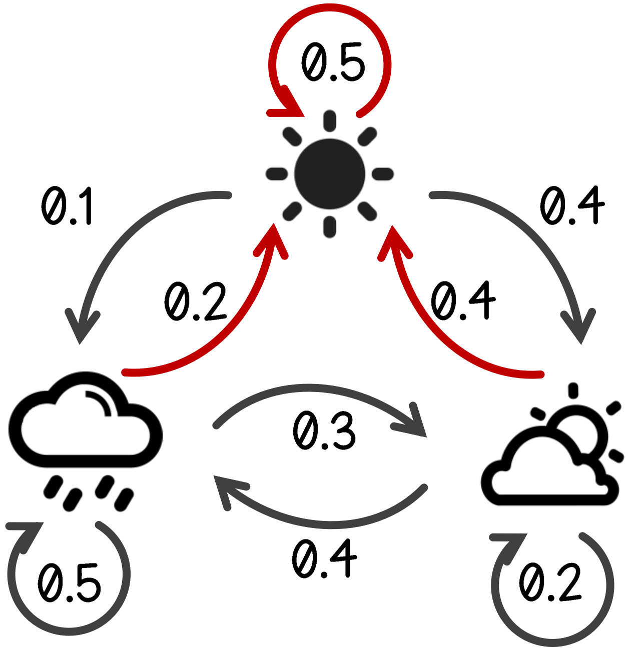
Markov Chain

Example: In a given city, if today's weather is sunny, there is a 40% chance that tomorrow will be partly cloudy, and a 10% chance of rain. If today's weather is partly cloudy, there is a 20% chance that tomorrow will also be partly cloudy, and a 40% chance of rain. If today's weather is rainy, there is a 50% chance that tomorrow will also be rainy, and a 30% chance of it being partly cloudy. Find the transition matrix of the problem.

W =

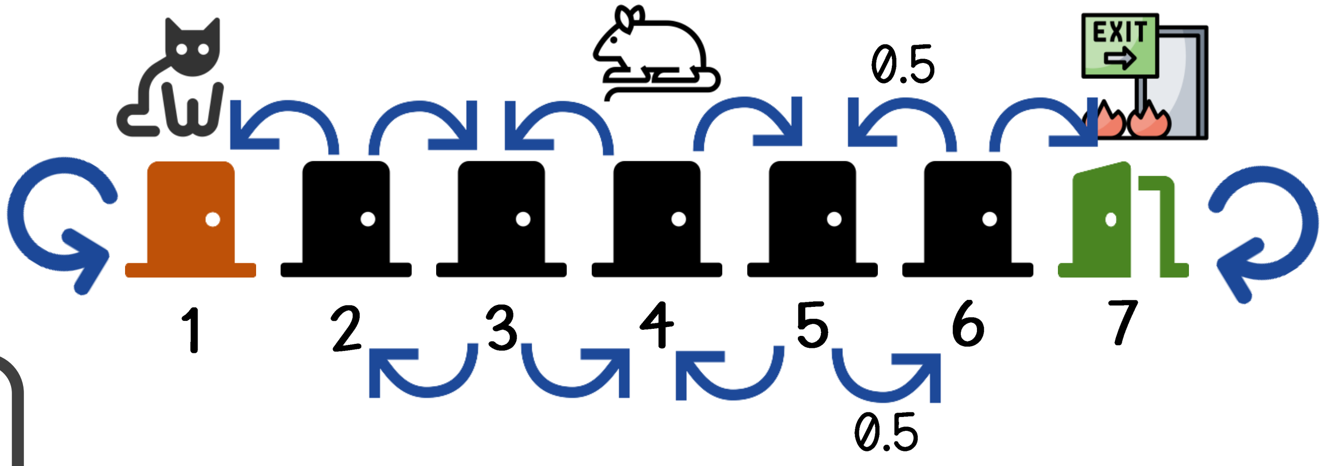
	S	C	R
S	0.5	0.4	0.1
C	0.4	0.2	0.4
R	0.2	0.3	0.5

$$W = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$



Markov Chain

Example: Find the transition matrix of the "**Cat & Mouse**" problem.

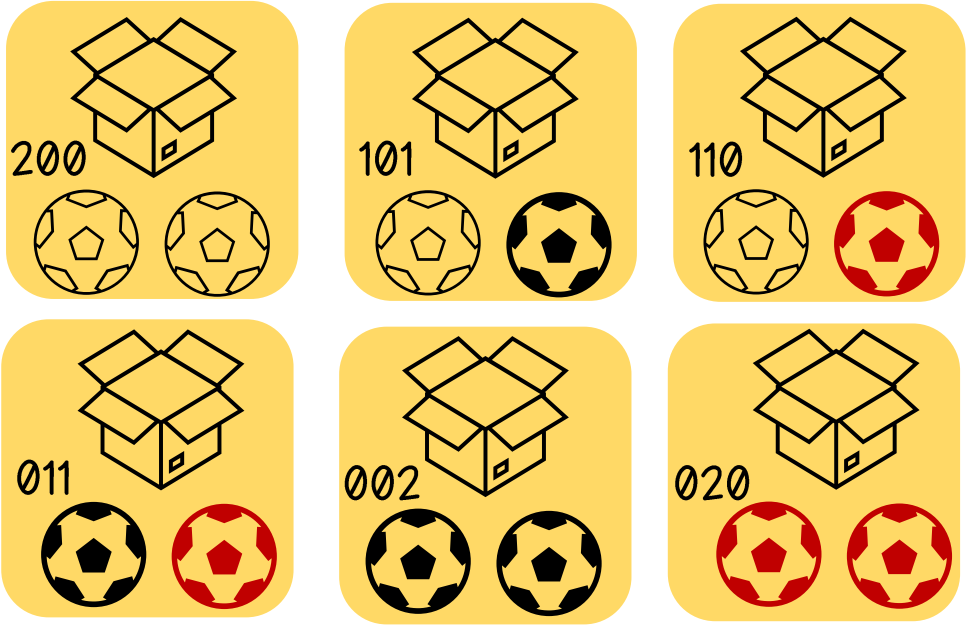


D =

	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	0.5	0	0.5	0	0	0	0
3	0	0.5	0	0.5	0	0	0
4	0	0	0.5	0	0.5	0	0
5	0	0	0	0.5	0	0.5	0
6	0	0	0	0	0.5	0	0.5
7	0	0	0	0	0	0	1

Markov Chain

Example: Find the transition matrix of the "**Balls from Urn**" problem.



$$
B = \begin{array}{c} \\ 200 \\ 110 \\ 101 \\ 011 \\ 020 \\ 002 \end{array}
\begin{array}{c} \begin{array}{cccccc} 200 & 110 & 101 & 011 & 020 & 002 \end{array} \\
\begin{bmatrix}
0 & 0.50 & 0.50 & 0 & 0 & 0 \\
0 & 0 & 0.50 & 0.25 & 0.25 & 0 \\
0 & 0.50 & 0 & 0.25 & 0 & 0.25 \\
0 & 0 & 0 & 0 & 0.50 & 0.50 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \end{array}
$$

Markov Chain

Example: Find the transition matrix of the "**Balls from Urn**" problem.

Whenever we select a ball from the box, we ensure that its color is altered. Consequently, no state is permitted to transition back to itself.

As any ball once painted cannot revert to an unpainted state, the transition probability for such a scenario is zero.

$$B = \begin{array}{c} \\ 200 \\ 110 \\ 101 \\ 011 \\ 020 \\ 002 \end{array} \begin{array}{c} \begin{array}{cccccc} 200 & 110 & 101 & 011 & 020 & 002 \end{array} \\ \begin{pmatrix} 0 & 0.50 & 0.50 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.25 & 0.25 & 0 \\ 0 & 0.50 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Markov Chain

Example: Find the transition matrix of the "**Balls from Urn**" problem.

Every time, the color of precisely one ball must undergo a change, making it impossible to simultaneously alter the colors of two balls.

If there are two painted balls of the same color in the box, it is certain that when we select one, its color must be changed.

B =

	200	110	101	011	020	002
200	0	0.50	0.50	0	0	0
110	0	0	0.50	0.25	0.25	0
101	0	0.50	0	0.25	0	0.25
011	0	0	0	0	0.50	0.50
020	0	0	0	1	0	0
002	0	0	0	1	0	0

Markov Chain

Example: Find the transition matrix of the "**Balls from Urn**" problem.

$$p(110|200) = p\begin{pmatrix}\text{unpainted} \\ \text{choosen}\end{pmatrix} \cap \begin{pmatrix}\text{getting} \\ \text{head}\end{pmatrix}$$

$$= 1 \times 0.5 = 0.5$$

$$p(101|110) = p\begin{pmatrix}\text{red} \\ \text{ball}\end{pmatrix} = 0.5$$

$$p(020|110) = p\begin{pmatrix}\text{unpainted} \\ \text{choosen}\end{pmatrix} \cap \begin{pmatrix}\text{getting} \\ \text{head}\end{pmatrix}$$

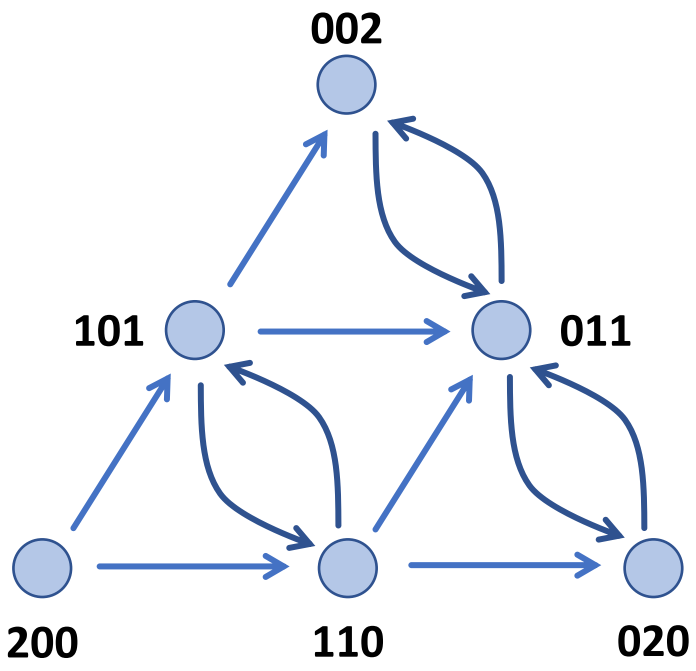
$$= 0.5 \times 0.5 = 0.25$$

B =

	200	110	101	011	020	002
200	0	0.50	0.50	0	0	0
110	0	0	0.50	0.25	0.25	0
101	0	0.50	0	0.25	0	0.25
011	0	0	0	0	0.50	0.50
020	0	0	0	1	0	0
002	0	0	0	1	0	0

Markov Chain

Example: Find the transition matrix of the "**Balls from Urn**" problem.



B =

	200	110	101	011	020	002
200	0	0.50	0.50	0	0	0
110	0	0	0.50	0.25	0.25	0
101	0	0.50	0	0.25	0	0.25
011	0	0	0	0	0.50	0.50
020	0	0	0	1	0	0
002	0	0	0	1	0	0

Course: Applied Probability

Chapter: [1]

Markov Chains

Section: [1.3]

n —Step Transition Probabilities

n-Step Transition

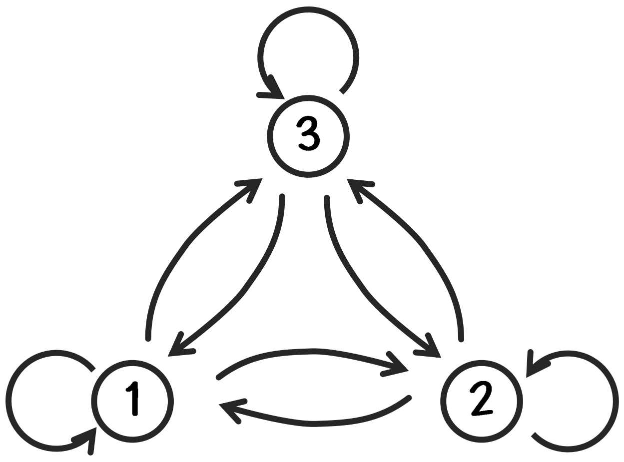
To understand the idea of this section, we start by an example.

Example: Consider the following Markov chain.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

* What is the probability of reaching **"state 3"** from **"state 2"** in exactly **ONE** step?

$$M_{23} = 0.4 = M_{23}(1)$$

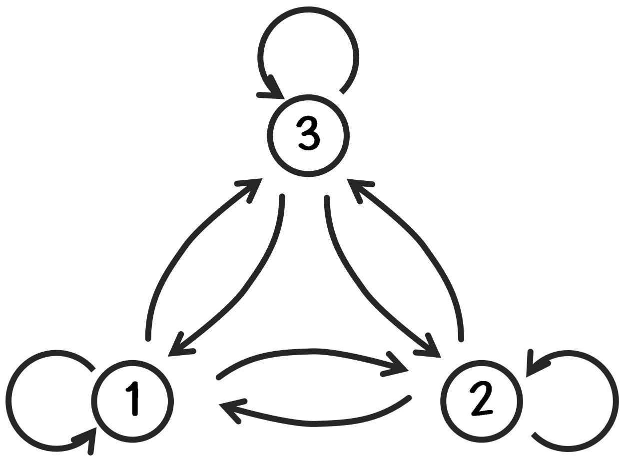


n-Step Transition

To understand the idea of this section, we start by an example.

Example: Consider the following Markov chain.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$



* What is the probability of reaching **"state 3"** from **"state 2"** in exactly **TWO** steps?

$$M_{23}(2) = M_{21}M_{13} + M_{22}M_{23} + M_{23}M_{33}$$

$$= (0.4)(0.1) + (0.2)(0.4) + (0.4)(0.5)$$

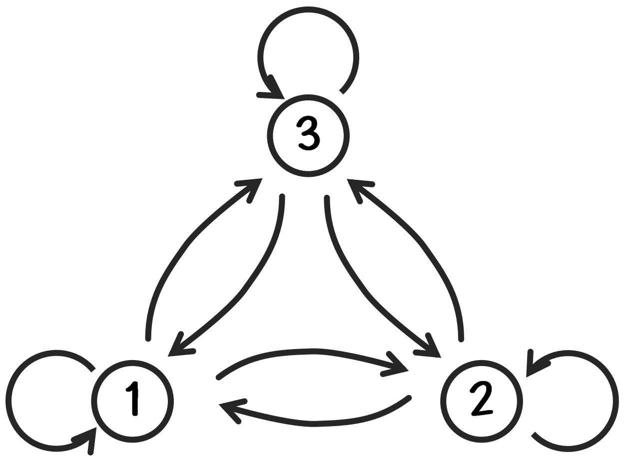
$$= 0.32$$

n-Step Transition

To understand the idea of this section, we start by an example.

Example: Consider the following Markov chain.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$



* What is the probability of reaching **"state 3"** from **"state 2"** in exactly **TWO** steps?

$$M_{23}(2) = M_{21}M_{13} + M_{22}M_{23} + M_{23}M_{33}$$

$$= [M_{21} \quad M_{22} \quad M_{23}] \cdot \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \end{bmatrix}$$

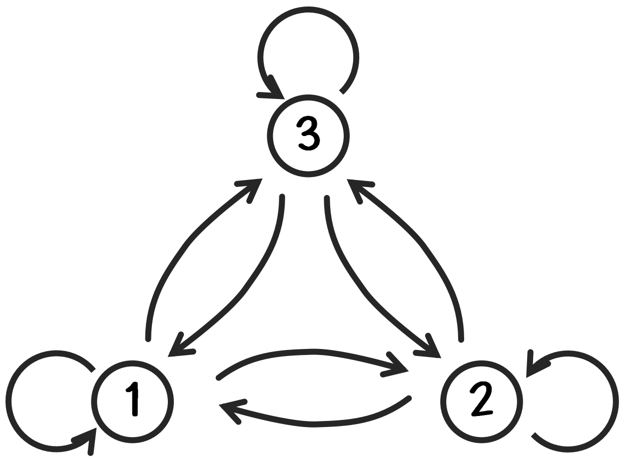
$$= [0.4 \quad 0.2 \quad 0.4] \cdot \begin{bmatrix} 0.1 \\ 0.4 \\ 0.5 \end{bmatrix} = 0.32$$

n-Step Transition

To understand the idea of this section, we start by an example.

Example: Consider the following Markov chain.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$



* What is the probability of reaching **"state j"** from **"state i"** in exactly **TWO** steps?

$$M_{ij}(2) = M^2_{ij} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.36 & 0.32 & 0.32 \\ 0.32 & 0.29 & 0.39 \end{bmatrix} \end{array}$$

$$M_{31}(2) = M^2_{31} = 0.32$$

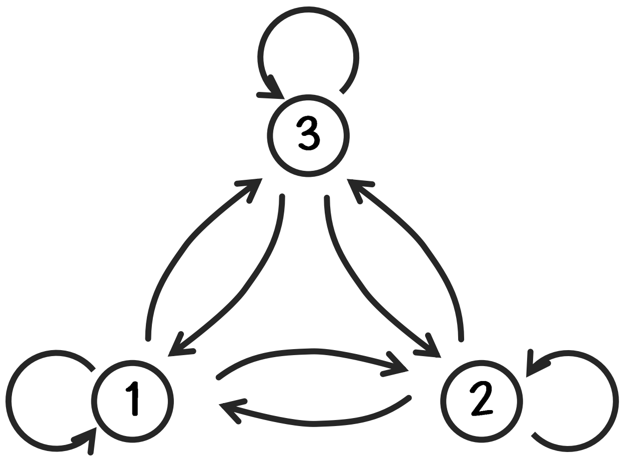
$$M_{11}(2) = M^2_{11} = 0.43$$

n-Step Transition

To understand the idea of this section, we start by an example.

Example: Consider the following Markov chain.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$



* What is the probability of reaching **"state j"** from **"state i"** in exactly **n**-steps?

$$M_{ij}(n) = M_{ij}^{n} = ij^{th} \text{ element of } M^n.$$

$$M^n = M\,M^{n-1} = M^{n-1}\,M$$

$$M^n = M^{n-k}\,M^k = M^k\,M^{n-k} \; ; \; 0 < k < n$$

n-Step Transition

Example: Back to the **weather** example. The Markov chain was

$$W = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{matrix} S & C & R \end{matrix} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

If it is rainy today, what is the probability that it will be sunny after 3-days?

$$W^3 = W^2 W = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.36 & 0.32 & 0.32 \\ 0.32 & 0.29 & 0.39 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{matrix} S & C & R \end{matrix} \\ \begin{bmatrix} 0.391 & 0.312 & 0.297 \\ 0.372 & 0.304 & 0.324 \\ 0.354 & 0.303 & 0.343 \end{bmatrix} \end{array}$$

$$\therefore W_{RS}(3) = 0.354$$

n-Step Transition

Example: Back to the **restaurant** example. The Markov chain was

$$M = \begin{array}{c} \\ B \\ Z \\ S \end{array} \begin{array}{c} \begin{matrix} B & Z & S \end{matrix} \\ \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{array}$$

If the restaurant serves Pizza today, what is the probability that it will serve Sausage after 4-days?

$$M^4 = M^2 M^2 = \begin{bmatrix} 0.30 & 0.14 & 0.56 \\ 0.34 & 0.25 & 0.41 \\ 0.31 & 0.29 & 0.40 \end{bmatrix} \begin{bmatrix} 0.30 & 0.14 & 0.56 \\ 0.34 & 0.25 & 0.41 \\ 0.31 & 0.29 & 0.40 \end{bmatrix} = \begin{array}{c} \\ B \\ Z \\ S \end{array} \begin{array}{c} \begin{matrix} B & Z & S \end{matrix} \\ \begin{bmatrix} .3112 & .2394 & .4494 \\ .3141 & .2290 & .4569 \\ .3156 & .2319 & .4525 \end{bmatrix} \end{array}$$

$$\therefore M_{ZS}(4) = 0.4569$$

The Probability Distribution of the States

Idea: In many situations, we **do not know the state of the Markov chain at time 0**.

In this case, we can determine the probability that the system is in "**state j**" at any "**time n**" by defining the vector

$$q = [q_1 \quad q_2 \quad \cdots \quad q_k]$$

where

q_1 is the probability of being at "**state 1**" at "**time 0**".

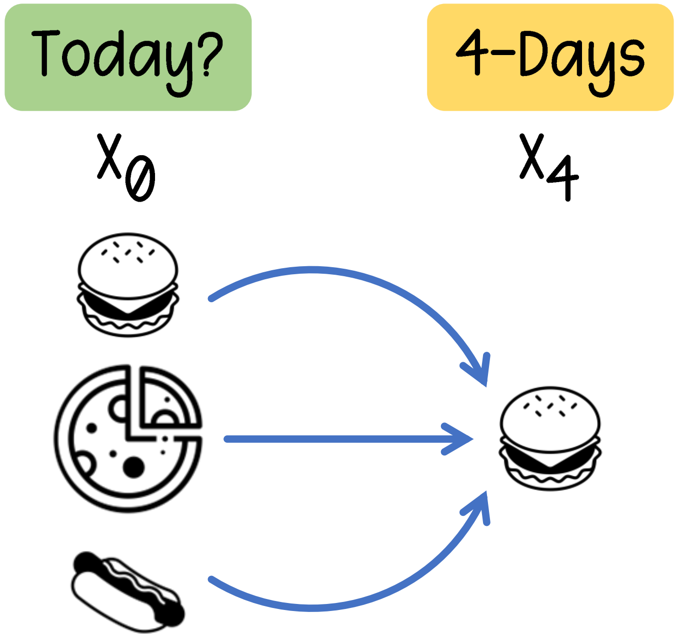
q_2 is the probability of being at "**state 2**" at "**time 0**".

$\vdots$

q_k is the probability of being at "**state s**" at "**time 0**".

The Probability Distribution of the States

Idea: In the restaurant example, someone asks: "**from today, what is the probability that the restaurant will serve Burger after 4-days?**"



$q_B = P(X_0 = B)$

$q_Z = P(X_0 = Z)$

$q_S = P(X_0 = S)$

$$\therefore M_B(4) = P(X_4 = B | X_0 = B)\,\boxed{P(X_0 = B)} + P(X_4 = B | X_0 = Z)\,\boxed{P(X_0 = Z)} + P(X_4 = B | X_0 = S)\,\boxed{P(X_0 = S)}$$

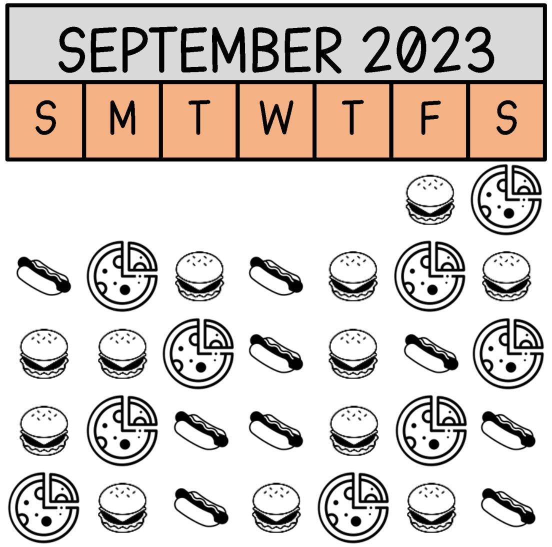
$$= M_{BB}(4)\, q_B + M_{ZB}(4)\, q_Z + M_{SB}(4)\, q_S$$

$$= \begin{bmatrix} q_B & q_Z & q_S \end{bmatrix} \begin{bmatrix} M_{BB}(4) \\ M_{ZB}(4) \\ M_{SB}(4) \end{bmatrix}$$

The Probability Distribution of the States

Idea: In the restaurant example, someone asks: "**from today, what is the probability that the restaurant will serve Burger after 4-days?**"

$$q_B \rightarrow \frac{12}{30} \qquad q_Z \rightarrow \frac{9}{30} \qquad q_S \rightarrow \frac{9}{30}$$



$$\therefore\ M_B(4) = \begin{bmatrix} q_B & q_Z & q_S \end{bmatrix} \begin{bmatrix} M_{BB}(4) \\ M_{ZB}(4) \\ M_{SB}(4) \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3115 \\ 0.3141 \\ 0.3156 \end{bmatrix}$$

$$= 0.31339$$

$$M^4 = \begin{array}{c} \\ B \\ Z \\ S \end{array} \begin{array}{c} \begin{array}{ccc} B & Z & S \end{array} \\ \begin{bmatrix} .3112 & .2394 & .4494 \\ .3141 & .2290 & .4569 \\ .3156 & .2319 & .4525 \end{bmatrix} \end{array}$$

The Probability Distribution of the States

Rule: Probability of being in state j at time n $= q \cdot \begin{bmatrix} \text{Column} \\ j \text{ of } M^n \end{bmatrix}$

Note: In the restaurant example,

$$q_0 M^4 = \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3112 & 0.2394 & 0.4494 \\ 0.3141 & 0.2290 & 0.4569 \\ 0.3156 & 0.2319 & 0.4525 \end{bmatrix}$$

$$= \begin{bmatrix} .31339 & .23403 & .45258 \end{bmatrix}$$ → Probability distribution of states after 4-days.

$$= q_4$$

The Probability Distribution of the States

Note: In the restaurant example, to calculate all the distribution probabilities of states for the first 4-days (**day-by-day**) from now:

$$q_0 = \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix}$$

$$q_1 = q_0 M$$
$$= \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.29 & 0.27 & 0.44 \end{bmatrix}$$

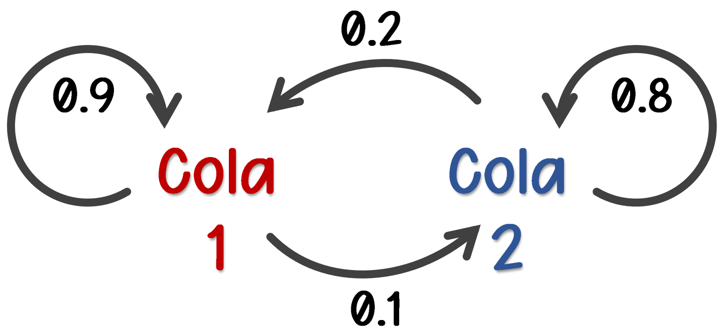
$$q_2 = q_1 M$$
$$= \begin{bmatrix} 0.29 & 0.27 & 0.44 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} .315 & .218 & .467 \end{bmatrix}$$

$$q_3 = q_2 M$$
$$= \begin{bmatrix} .315 & .218 & .467 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} .3152 & .2357 & .4491 \end{bmatrix}$$

$$q_4 = q_3 M$$
$$= \begin{bmatrix} .3152 & .2357 & .4491 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} .31339 & .23403 & .45258 \end{bmatrix}$$

The Probability Distribution of the States

Example: Suppose the entire cola industry produces only two colas. Given that a person last purchased cola 1, there is a 90% chance that her next purchase will be cola 1. Given that a person last purchased cola 2, there is an 80% chance that her next purchase will be cola 2.



1. Write the transition matrix of the model.

$$C = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

The Probability Distribution of the States

Example: Suppose the entire cola industry produces only two colas. Given that a person last purchased cola 1, there is a 90% chance that her next purchase will be cola 1. Given that a person last purchased cola 2, there is an 80% chance that her next purchase will be cola 2.

2. If a person is **currently** a cola 1 purchaser, what is the probability that she will purchase cola 1 **two** purchases from now?

$$C = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

$$C^2 = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \end{array}$$

$$C_{11}(2) = C^2_{11} = 0.83$$

The Probability Distribution of the States

Example: Suppose the entire cola industry produces only two colas. Given that a person last purchased cola 1, there is a 90% chance that her next purchase will be cola 1. Given that a person last purchased cola 2, there is an 80% chance that her next purchase will be cola 2.

3. Suppose **60%** of all people **now** drink cola 1, and **40% now** drink cola 2. **Two** purchases from now, what fraction of all purchasers will be drinking cola 1? will be drinking cola 2?

$$C = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

$$q_2 = q_0 C^2 = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

$$= \begin{bmatrix} .634 & .366 \end{bmatrix}$$

Cola 1 ← .634 .366 → Cola 2

$$C^2 = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \end{array}$$

Course: Applied Probability

Chapter: [1]
Markov Chains

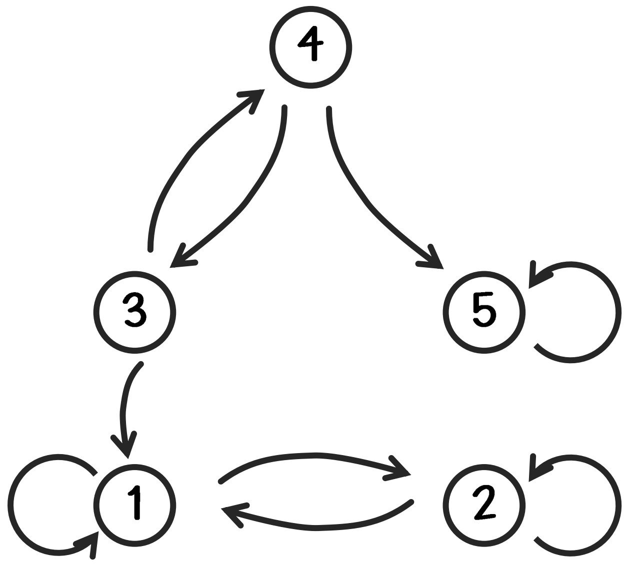
Section: [1.4]
Classification of States in a Markov Chain

Classifications of States

Knowing the classification of the states of the stochastic system, and what type of different states we have, is important to be able to talk about behavior of stochastic systems in the **long run**.

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0.4 & 0.6 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0.7 & 0 \\
0 & 0 & 0.9 & 0 & 0.1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

The following example is used to understand the first part of definitions.



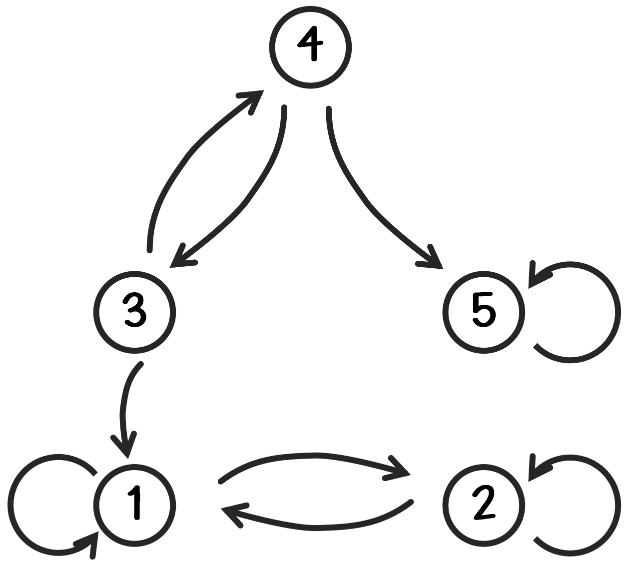
Classifications of States

Definition (1)

Given two states i and j, a "**path**" from i to j is a sequence of transitions that begins in i and ends in j, such that each transition in the sequence has a positive probability of occurring.

P =

	1	2	3	4	5
1	0.4	0.6	0	0	0
2	0.5	0.5	0	0	0
3	0.3	0	0	0.7	0
4	0	0	0.9	0	0.1
5	0	0	0	0	1



For example, 3–4–5 is a path from 3 to 5.

$$P^2_{35} > 0$$

Note that $P_{35} = 0$

Classifications of States

Definition (2)

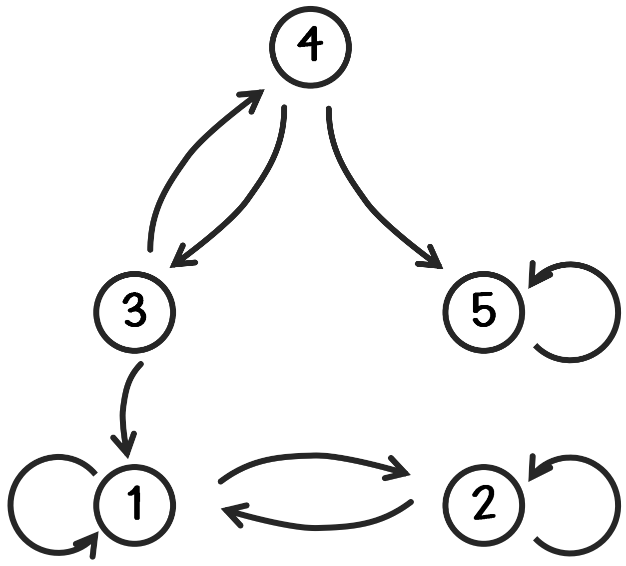
A state j is **"reachable"** from state i if there is a path leading from i to j.

$$P = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0.4 & 0.6 & 0 & 0 & 0 \\ 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 3 & 0.3 & 0 & 0 & 0.7 & 0 \\ 4 & 0 & 0 & 0.9 & 0 & 0.1 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

For example,

* state 2 is reachable from state 4 via the path 4-3-1-2,

* but state 4 is not reachable from state 2 since there is no path from 2 to 4.



Classifications of States

Definition (3)

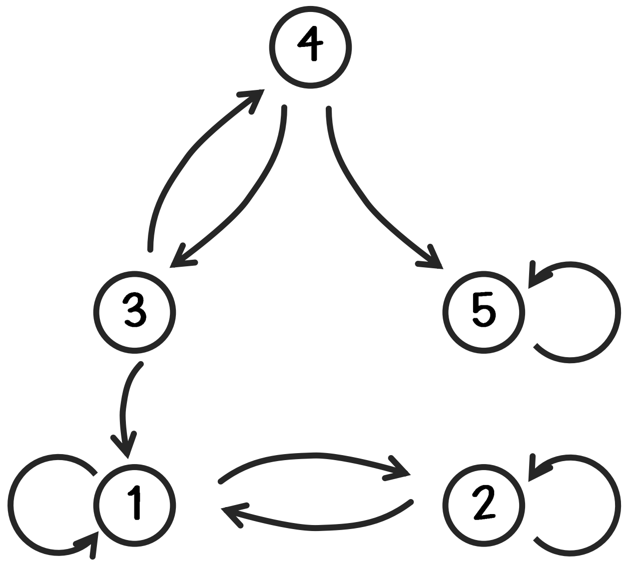
Two states i and j are said to "**communicate**" if j is reachable from i, and i is reachable from j.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

For example,

* States 1 and 2 are communicate since we can go from 1 to 2 and from 2 to 1.

* States 3 and 2 are not communicate since state 2 is reachable from state 3, but state 3 is not reachable from state 2.



Classifications of States

Definition (4)

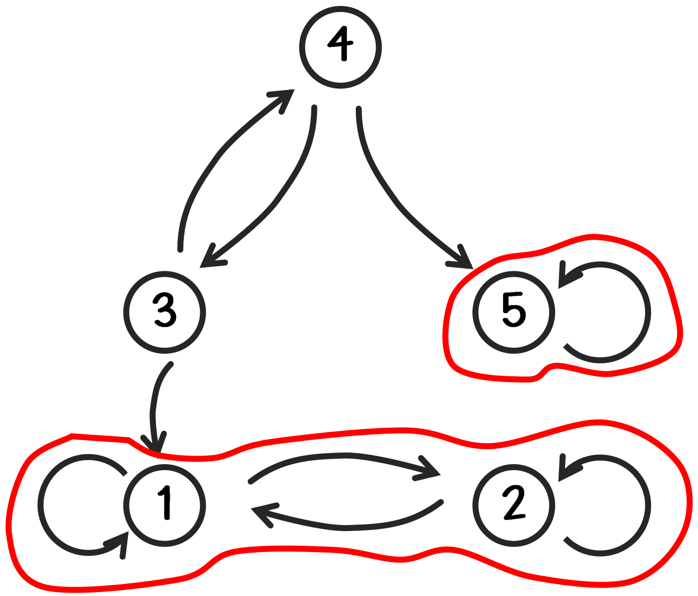
A set of states S in a Markov chain is a "**closed set**" if no state outside of S is reachable from any state in S.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

For example,

* The set $S_1 = \{1,2\}$ is closed set.

* The set $S_2 = \{5\}$ is closed set.



Classifications of States

Definition (5)

A state i is an "**absorbing**" state if $p_{ii} = 1$.

For example, state 5 is the only absorbing state.

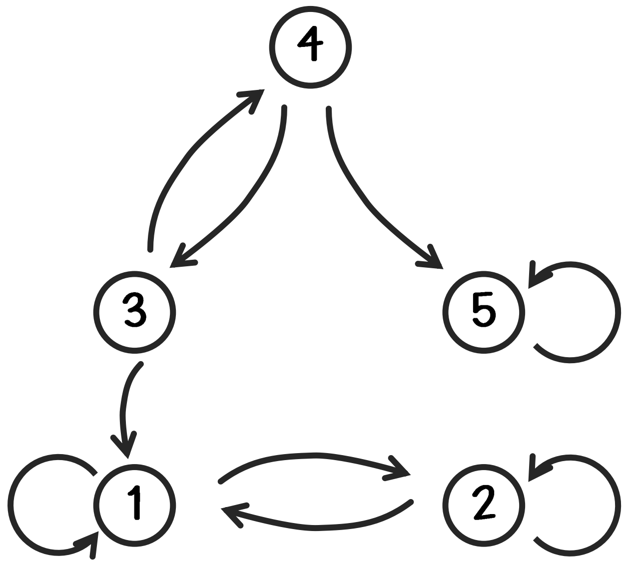
$P =$

	1	2	3	4	5
1	0.4	0.6	0	0	0
2	0.5	0.5	0	0	0
3	0.3	0	0	0.7	0
4	0	0	0.9	0	0.1
5	0	0	0	0	1

Notes

* Whenever we enter an absorbing state, we never leave the state.

* An absorbing state is a closed set containing only one state.



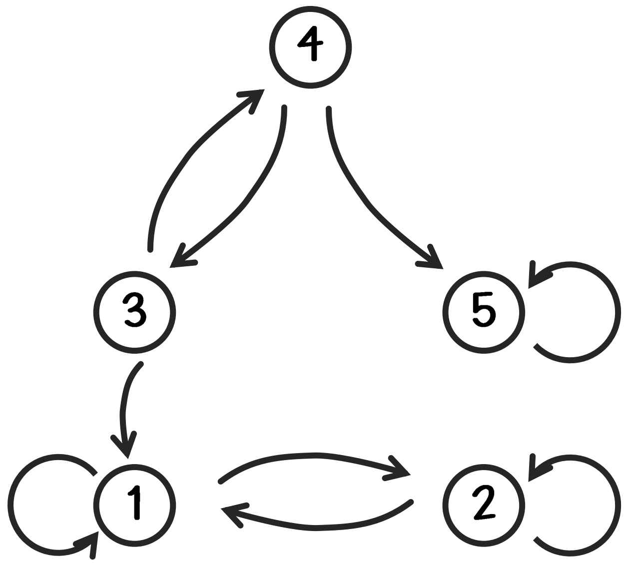
Classifications of States

Definition (6)

A state i is a **"transient"** state if **there exists** a state j that is reachable from i, but the state i is not reachable from state j.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

* In other words, a state i is **"transient"** if there is a way to leave state i that never returns to state i.



Classifications of States

State 3 is Transient

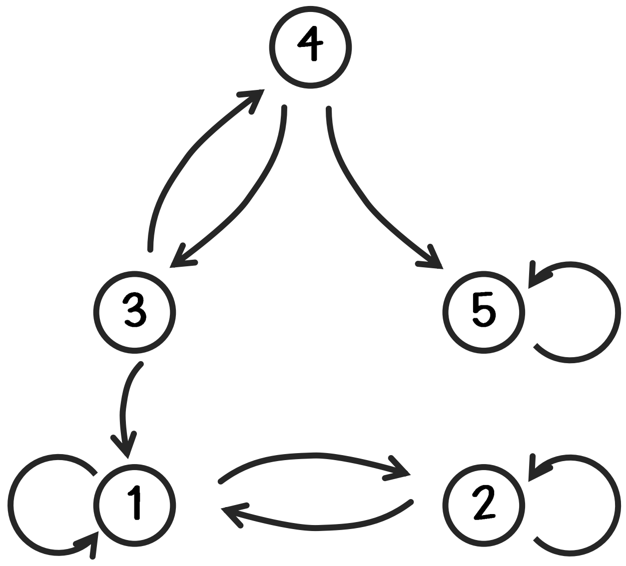
Because we can go from state 3 to state 1, but we cannot go back to state 3 from state 1.

State 4 is Transient

Because we can go from state 4 to state 5, but we cannot go back to state 4 from state 5.

Note: After a large number of periods, the probability of being in any transient state is **0**.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



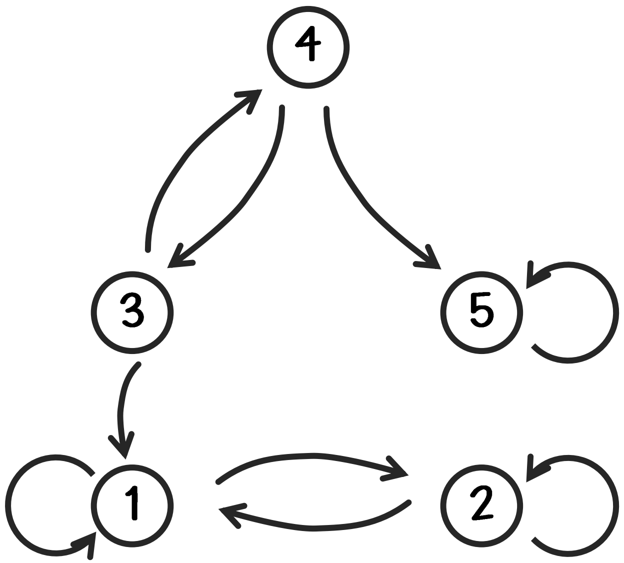
Classifications of States

Definition (7)

If a state is not transient, it is called a **"recurrent"** state.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

States 1, 2, and 5 are recurrent states.



Note: Every absorbing state is a recurrent state, but not every recurrent state is an absorbing state.

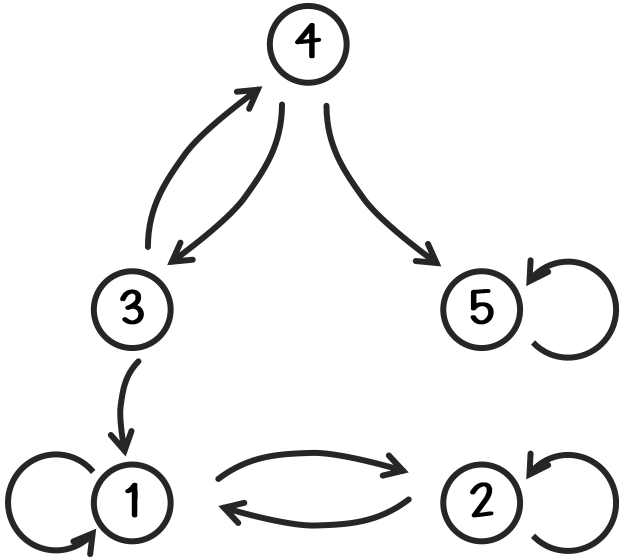
Classifications of States

Definition (8)

A state i is "**periodic**" with period $k > 1$ if k is the smallest number such that all paths leading from state i back to state i have a length that is a multiple of k. If a recurrent state is not periodic, it is referred to as "**aperiodic**".

$P =$

	1	2	3	4	5
1	0.4	0.6	0	0	0
2	0.5	0.5	0	0	0
3	0.3	0	0	0.7	0
4	0	0	0.9	0	0.1
5	0	0	0	0	1



The states 1, 2, and 5 are aperiodic

$k =$ length of the shortest path from state i to itself

$= 1 \ngtr 1$

Classifications of States

State 3 is periodic with period 2

The shortest path from state 3 back to state 3 is 3-4-3

$k = 2 > 1$

Another path is 3-4-3-4-3

$L = 4 = 2k$

State 4 is periodic with period 2

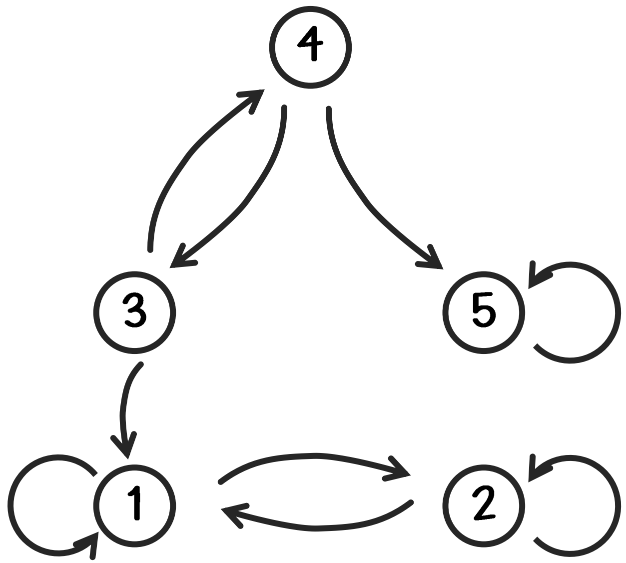
The shortest path from state 4 back to state 4 is 4-3-4

$k = 2 > 1$

Another path is 4-3-4-3-4

$L = 4 = 2k$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Classifications of States

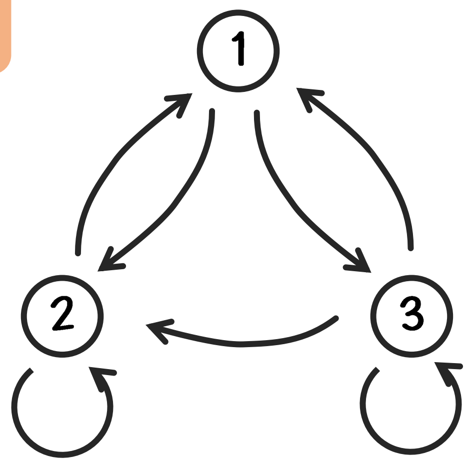
Definition (9)

If **all** states in a chain are **recurrent**, **aperiodic**, and **communicate** with each other, the chain is said to be "**ergodic**".

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.3 & 0.7 & 0 \\ 0.4 & 0.5 & 0.1 \end{bmatrix} \end{array}$$

State	Recurrent?	Aperiodic?
1	Yes	Yes
2	Yes	Yes
3	Yes	Yes
	✓	✓

States	Communicate?
1 , 2	Yes
1 , 3	Yes
2 , 3	Yes
	✓



Ergodic

Course: Applied Probability

Chapter: [1]

Markov Chains

Section: [1.5]

Steady-State Probabilities and Mean First Passage Times

Steady-State Probabilities

Idea To illustrate the behavior of the n-step transition probabilities for **large values** of n, we have computed several of the n-step transition probabilities for the Cola example (Section 1.3) as follows:

$$C^{1} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array} \qquad C^{4} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.75 & 0.25 \\ 0.51 & 0.49 \end{bmatrix} \end{array} \qquad C^{20} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \end{array}$$

$$C^{2} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \end{array} \qquad C^{5} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix} \end{array} \qquad C^{30} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \end{array}$$

$$C^{3} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.78 & 0.22 \\ 0.44 & 0.56 \end{bmatrix} \end{array} \qquad C^{10} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.68 & 0.32 \\ 0.65 & 0.35 \end{bmatrix} \end{array} \qquad C^{40} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \end{array}$$

Steady-State Probabilities

Idea

$$\lim_{n\to\infty} C^n = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \end{array}$$

* After a long time, the probability that a person's next cola purchase would be cola 1 approached .67 and .33 that it would be cola 2.

* These probabilities did not depend on whether the person was initially a cola 1 or a cola 2 drinker.

* For large n, the matrix C^n approaches a matrix with identical rows. This means that after a long time, the Markov chain settles down, and there is a probability π_j that we are in state j.

$$[\pi_1 \quad \pi_2] = [0.67 \quad 0.33]$$

Steady-State Probabilities

Note The vector $\pi = [\pi_1 \quad \pi_2 \quad \cdots \quad \pi_k]$ is often called the "**steady-state distribution**", or "**equilibrium distribution**", for the Markov chain.

Question Does every Markov chain reach the steady-state distribution?

Theorem Let P be the transition matrix for k—state **ergodic chain**. Then there exists a vector $\pi = [\pi_1 \quad \pi_2 \quad \cdots \quad \pi_k]$ such that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \\ \pi_1 & \pi_2 & \cdots & \pi_k \\ & & \vdots & \\ \pi_1 & \pi_2 & \cdots & \pi_k \end{bmatrix} \quad ; \quad \pi_1 + \pi_2 + \cdots + \pi_k = 1$$

Steady-State Probabilities

Question How we can find the steady-state distribution for an ergodic Markov chain?

For large n, we have $P_{ij}(n+1) \cong P_{ij}(n) \cong \pi_j$

$$\pi_j = P_{ij}(n+1) = P_{ij}^{n+1} = P_{ij}^{n} P_{ij}$$

$$= \left[i^{th} \text{ row of } P^n\right] \cdot \left[\begin{matrix} j^{th} \text{ column} \\ \text{of } P \end{matrix}\right]$$

$$= \pi_1 p_{1j} + \pi_2 p_{2j} + \cdots + \pi_k p_{kj}$$

$$P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \\ & \vdots & & \\ \pi_1 & \pi_2 & \cdots & \pi_k \\ & \vdots & & \\ \pi_1 & \pi_2 & \cdots & \pi_k \end{bmatrix}$$

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1k} \\ p_{21} & \cdots & p_{2j} & \cdots & p_{2k} \\ \vdots & & \vdots & & \vdots \\ p_{k1} & \cdots & p_{kj} & \cdots & p_{kk} \end{bmatrix}$$

In general, for large n, we have $\pi = \pi P$

Steady-State Probabilities

Question How we can find the steady-state distribution for an ergodic Markov chain?

* If n is large, solve the system $\pi = \pi P$.

* Unfortunately, this system of equations has an infinite number of solutions.

* Replace any equation in $\pi = \pi P$ by $\pi_1 + \pi_2 + \cdots + \pi_k = 1$.

Steady-State Probabilities

Example Find the steady-state probabilities for the cola example.

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\pi = \pi P$$

$$[\pi_1 \quad \pi_2] = [\pi_1 \quad \pi_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\pi_1 = 0.9\pi_1 + 0.2\pi_2 \longrightarrow 0.1\pi_1 = 0.2\pi_2$$

$$\pi_2 = 0.1\pi_1 + 0.8\pi_2 \longrightarrow 0.2\pi_2 = 0.1\pi_1$$

$$\pi_1 + \pi_2 = 1$$

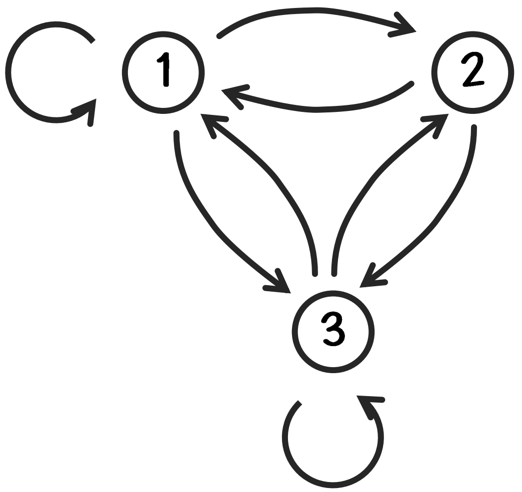
$$\left.\begin{aligned} \pi_1 &= 2\pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}\right\} \begin{aligned} \pi_1 &= 2/3 \\ \pi_2 &= 1/3 \end{aligned}$$

Steady-State Probabilities

Example Consider the following Markov chain:

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

1 Show that the chain is **ergodic**.

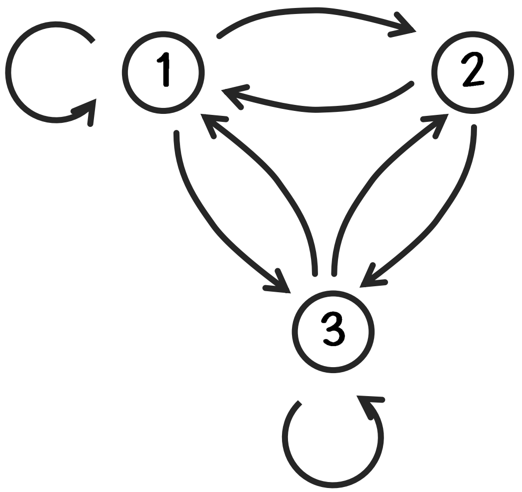


State	Recurrent?	Aperiodic?	Communicate?
1	Yes	Yes	Yes with 2, 3
2	Yes	Yes	Yes with 1, 3
3	Yes	Yes	Yes with 1, 2

Steady-State Probabilities

Example Consider the following Markov chain:

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$



2 Find the **steady-state distribution** of the chain.

$$\pi = \pi P$$

$$[\pi_1 \quad \pi_2 \quad \pi_3] = [\pi_1 \quad \pi_2 \quad \pi_3]\begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

$$\pi_1 = 0.2\pi_1 + 0.3\pi_2 + 0.4\pi_3$$

$$\pi_2 = 0.6\pi_1 + 0.1\pi_3$$

$$\pi_3 = 0.2\pi_1 + 0.7\pi_2 + 0.5\pi_3$$

Replace by $\pi_1 + \pi_2 + \pi_3 = 1$

$$-0.8\pi_1 + 0.3\pi_2 + 0.4\pi_3 = 0$$

$$0.6\pi_1 - \pi_2 + 0.1\pi_3 = 0$$

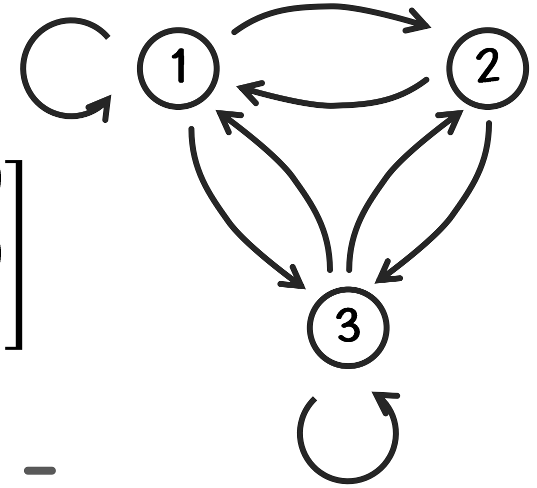
$$\pi_1 + \pi_2 + \pi_3 = 1$$

Steady-State Probabilities

Example Consider the following Markov chain:

$$P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

2 Find the **steady-state distribution** of the chain.



$$-8\pi_1 + 3\pi_2 + 4\pi_3 = 0$$
$$6\pi_1 - 10\pi_2 + \pi_3 = 0$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

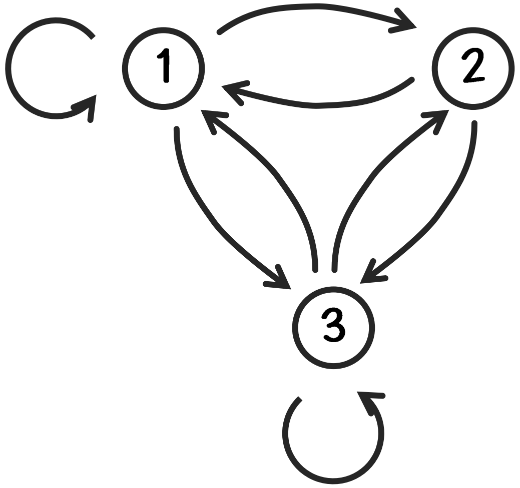
$$\begin{bmatrix} -8 & 3 & 4 \\ 6 & -10 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} -8 & 3 & 4 \\ 6 & -10 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -11/137 & 1/137 & 43/137 \\ -5/137 & -12/137 & 32/137 \\ 16/137 & 11/137 & 62/137 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 43/137 \\ 32/137 \\ 62/137 \end{bmatrix}$$

Steady-State Probabilities

Example Consider the following Markov chain:

$$P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$



2 Find the **steady-state distribution** of the chain.

$$[\pi_1 \quad \pi_2 \quad \pi_3] = [.31387 \quad .23358 \quad .45255]$$

Mean First Passage Times

Idea

Imagine you're playing a board game where you move from one square to another based on the roll of a die, and you want to know, **on average, how many rolls it will take to reach a particular square for the first time?**

The **"Mean First Passage Time"** for an <u>ergodic</u> Markov chain, represents the expected number of steps it would take for the system to reach a particular state "j" starting from a given initial state "i", and is denoted by m_{ij}.

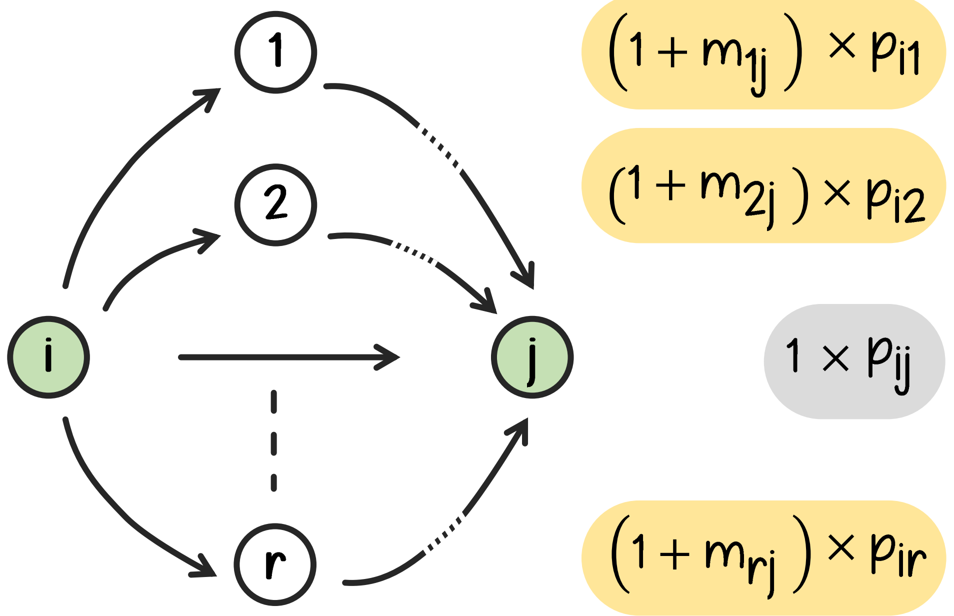
Mean First Passage Times

Formulas

$$m_{ij} = p_{ij} + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik}(1 + m_{kj})$$

$$= p_{ij} + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj}$$

$$= 1 + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj}$$



$$m_{ij} = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj}$$

$$m_{jj} = 1/\pi_j$$

Mean First Passage Times

$$m_{ij} = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] = [2/3 \quad 1/3]$$

Example Find all the mean first passage times for all the states in the **Cola** example.

$$m_{11} = \frac{1}{\pi_1} = \frac{1}{2/3} = \frac{3}{2}$$

The person who last drank cola 1 will drink, on average, bottle and half of cola before drinking cola 1 again.

$$m_{22} = \frac{1}{\pi_2} = \frac{1}{1/3} = 3$$

The person who last drank cola 2 will drink, on average, 3 bottles of cola before drinking cola 2 again.

Mean First Passage Times

$$m_{ij} = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] = [2/3 \quad 1/3]$$

Example Find all the mean first passage times for all the states in the **Cola** example.

$$m_{12} = 1 + \sum_{\substack{k=1 \\ k \neq 2}}^{2} p_{1k} m_{k2} = 1 + p_{11} m_{12}$$

$$m_{12} = 1 + 0.9 m_{12}$$

$$0.1\, m_{12} = 1 \qquad \therefore m_{12} = 10$$

The person who last drank cola 1 will drink, on average, 10 bottles of cola before switching to cola 2.

Mean First Passage Times

$$m_{ij} = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{r} p_{ik} m_{kj} \qquad m_{ii} = 1/\pi_i$$

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] = [2/3 \quad 1/3]$$

Example Find all the mean first passage times for all the states in the **Cola** example.

$$m_{21} = 1 + \sum_{\substack{k=1 \\ k \neq 1}}^{2} p_{2k} m_{k1} = 1 + p_{22} m_{21}$$

$$m_{21} = 1 + 0.8 m_{21}$$

$$0.2\, m_{21} = 1 \qquad \therefore m_{21} = 5$$

The person who last drank cola 2 will drink, on average, 5 bottles of cola before switching to cola 1.

Course: Applied Probability

Chapter: [1]
Markov Chains

Section: [1.6]
Absorbing Chains

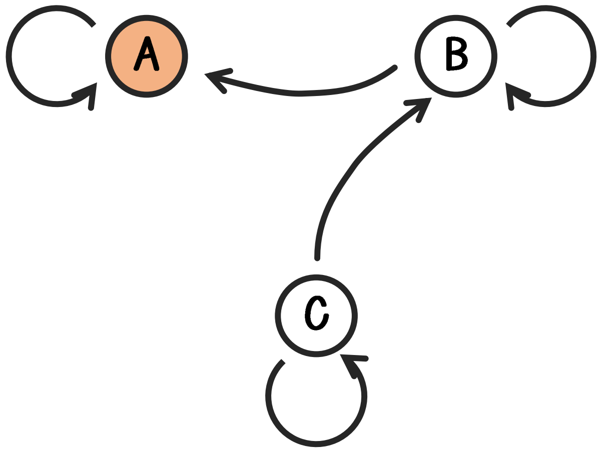
Absorbing Markov Chain

Remember A state **i** in a Markov chain is called an **absorbing state** if $p_{ii} = 1$.

$$P = \begin{matrix} A \\ B \\ C \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

State "A" is an absorbing state because $p_{AA} = 1$.

States B and C are nonabsorbing states.



Once the absorbing state is entered, it is impossible to leave.

Absorbing Markov Chain

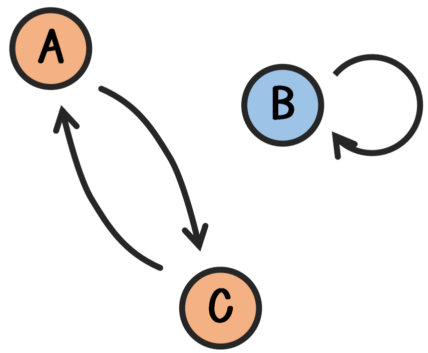
Definition A Markov chain is an "**absorbing chain**" if

1) There is at least one absorbing state; and
2) it is possible to go from each nonabsorbing state to at least one absorbing state in a finite number of steps.

Example Determine whether the following Markov chain is absorbing.

$$P = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Although state B is an absorbing state, it is impossible to go from either state A or state C to the absorbing state B. So, the given Markov chain is not an absorbing Markov chain.



Absorbing Markov Chain

Questions

For any absorbing chain, one might want to know certain things.

1) If the chain begins in a given transient state, and before we reach an absorbing state,
   - what is the expected number of times that each state will be entered?
   - How many periods do we expect to spend in a given transient state before absorption takes place?

2) If a chain begins in a given transient state, what is the probability that we end up in each absorbing state?

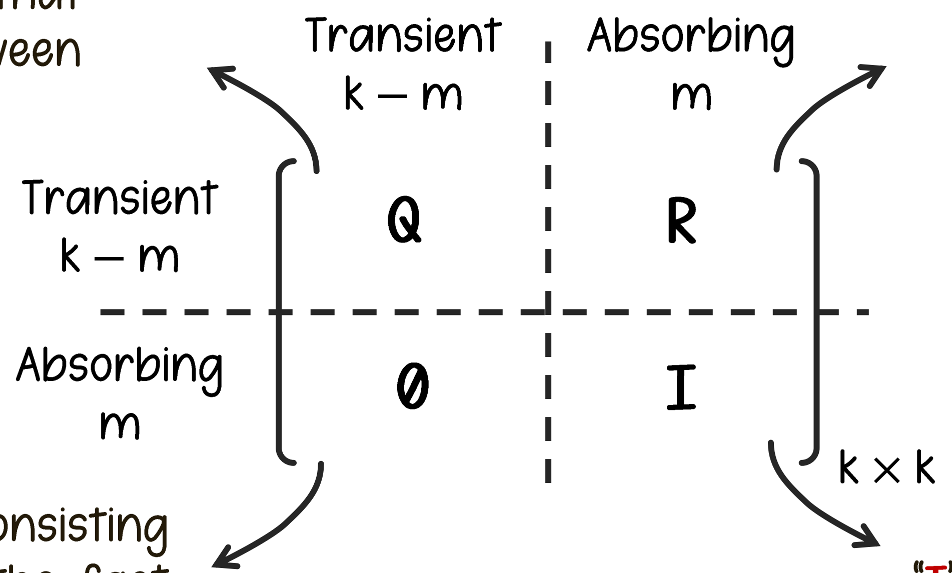
Absorbing Markov Chain

Standard Form

To answer those questions, we need to write the transition matrix in **"standard form"** where the states in this form listed in the following order.

"Q" is an (k-m)×(k-m) matrix that represents transitions between transient states.

"R" is an (k-m)×m matrix representing transitions from transient states to absorbing states.



"0" is an m×(k-m) matrix consisting entirely of zeros. This reflects the fact that it is impossible to go from an absorbing state to a transient state.

"I" is an m×m identity matrix reflecting the fact that we can never leave an absorbing state.

Absorbing Markov Chain

Example Write the following transition matrix as standard form.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} .2 & .3 & .4 & .1 \\ 0 & 1 & 0 & 0 \\ .5 & .3 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$P = \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} .2 & .3 & .4 & .1 \\ .5 & .3 & 0 & .2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$P = \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\ \left(\begin{array}{cc|cc} .2 & .4 & .3 & .1 \\ .5 & 0 & .3 & .2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

Q R

Absorbing Markov Chain

Goal Given the matrices R and Q, and the unit column vector $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, then

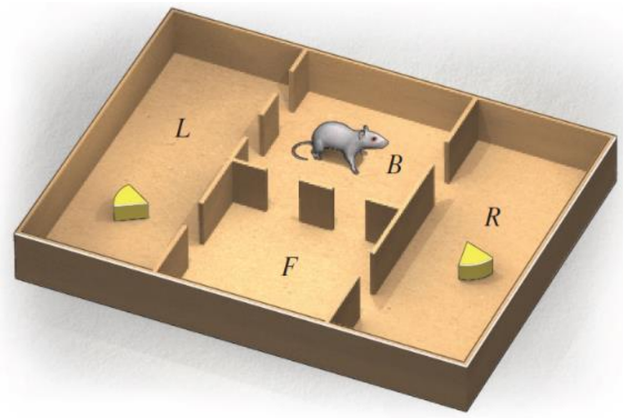
Expected time in state j starting in state i $\longrightarrow$ ij^{th} element of $(I-Q)^{-1}$.

Expected time to absorption $\longrightarrow$ $(I-Q)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Probability of absorption $\longrightarrow$ $(I-Q)^{-1} R$

Absorbing Markov Chain

Example A rat is placed in room F or room B of the maze shown in the figure. The rat wanders from room to room until it enters one of the rooms containing food, L or R. Assume that the rat chooses an exit from a room at random and that once it enters a room with food it never leaves.



a) Find the transition matrix of the problem.

$$Q = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} b & f \end{array} \\ \begin{bmatrix} 0 & .4 \\ .5 & 0 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} .4 & .2 \\ .25 & .25 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} \\ b \\ f \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cccc} b & f & l & r \end{array} \\ \left[\begin{array}{cc|cc} 0 & .4 & .4 & .2 \\ .5 & 0 & .25 & .25 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Absorbing Markov Chain

Example b) What is the long-run probability that a rat placed in room **b** ends up in room **r**? → .375

$$P = \begin{array}{c} \\ b \\ f \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cccc} b & f & l & r \end{array} \\ \left[\begin{array}{cc|cc} 0 & .4 & .4 & .2 \\ .5 & 0 & .25 & .25 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$Q = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} b & f \end{array} \\ \begin{bmatrix} 0 & .4 \\ .5 & 0 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} .4 & .2 \\ .25 & .25 \end{bmatrix} \end{array}$$

$$I-Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .4 \\ .5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.4 \\ -.5 & 1 \end{bmatrix}$$

$$(I-Q)^{-1} = \frac{1}{(1)(1)-(-.4)(-.5)} \begin{bmatrix} 1 & .4 \\ .5 & 1 \end{bmatrix} = \begin{bmatrix} 1.25 & .5 \\ .625 & 1.25 \end{bmatrix}$$

$$(I-Q)^{-1}R = \begin{array}{c} b \\ f \end{array} \begin{bmatrix} 1.25 & .5 \\ .625 & 1.25 \end{bmatrix} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} .4 & .2 \\ .25 & .25 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} .6250 & .3750 \\ .5625 & .4375 \end{bmatrix} \end{array}$$

Absorbing Markov Chain

Example c) What is the average number of exits that a rat placed in room **b** will choose until it finds food?

→ 1.75

$$P = \begin{array}{c} \\ b \\ f \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cccc} b & f & l & r \end{array} \\ \left[\begin{array}{cc|cc} 0 & .4 & .4 & .2 \\ .5 & 0 & .25 & .25 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$Q = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} b & f \end{array} \\ \begin{bmatrix} 0 & .4 \\ .5 & 0 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \\ b \\ f \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} .4 & .2 \\ .25 & .25 \end{bmatrix} \end{array}$$

$$(I-Q)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{array}{c} b \\ f \end{array} \begin{bmatrix} 1.25 & .5 \\ .625 & 1.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.750 \\ 1.875 \end{bmatrix}$$