

# Linear Programming

## Lecture Notes for Math 373

Feras Awad

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# 1 Introduction to Linear Programming

## 1.1 Operations Researches

The first formal activity of operations research (OR) were initiated in England during World War II. A team of British scientists set out to make decisions regarding the best utilization of war materials. After war, the idea was adopted and improved in the civilian sector.

In OR, we do not have a single general technique to solve all mathematical models. The most OR techniques are: Linear Programming, Non-Linear Programming, Integer Programming, Dynamic Programming, Network Programming, and much more. All techniques are determined by algorithms, and not by closed form formulas. The algorithms are for deterministic models (not for probability or stochastic model). Deterministic means when we know the values of the variables we know the value of the model in certainty.

## 1.2 What is a Linear Programming (LP) Problem?

**Definition 1.1.** A function  $f(x_1, x_2, \dots, x_n)$  of the variables  $x_1, x_2, \dots, x_n$  is a *linear function* if  $f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n$  where  $c_1, c_2, \dots, c_n$  are constants.

**Example 1.1.**  $f(x_1, x_2) = 2x_1 + x_2$  is linear function, while  $f(x_1, x_2) = x_1^2x_2$  is not linear.

**Definition 1.2.** For any linear function  $f(x_1, x_2, \dots, x_n)$  and any real number  $b$ , the inequalities  $f(x_1, x_2, \dots, x_n) \leq b$  and  $f(x_1, x_2, \dots, x_n) \geq b$  are *linear inequalities*.

**Example 1.2.**  $2x_1 + 3x_2 \leq 3$  is linear inequality, while  $x_1x_2 + \sqrt{x_2} \geq 3$  is not linear inequality.

**Definition 1.3.** A *linear programming (LP) problem* is an optimization problem for which we do the following

1. We attempt to *maximize (profit)* or *minimize (cost)* a linear function (called the *objective function*) of the *decision variables*.
2. The values of the decision variables must satisfy a set of *constraints*, and each constraint must be linear equation or linear inequality.

3. A *sign restriction* is associated with each variable. For any variable  $x_i$ , either  $x_i \geq 0$  or  $x_i$  is unrestricted in sign (urs).

$$\begin{array}{ll} \text{max (or min)} & z = f(x_1, x_2, \dots, x_n) \\ \text{Subject to} & \text{Constraints} \\ & \text{Sign Restriction.} \end{array}$$

**Example 1.3.** Furnco manufactures desks and chairs. Each desk uses 4 units of wood, and each chair uses 3. A desk contributes \$40 to profit, and a chair contributes \$25. Marketing restrictions require that the number of chairs produced be at least twice the number of desks produced. If 20 units of wood are available, formulate an LP to maximize Furnco's profit.



**Solution:** Let  $x_1$  be the number of desks produced, and  $x_2$  be the number of chairs produced. Then, the formulation of the problem is

$$\begin{array}{ll} \text{max} & z = 40x_1 + 25x_2 \\ \text{s.t.} & 4x_1 + 3x_2 \leq 20 \\ & x_2 \geq 2x_1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

The optimal solution of this problem is  $x_1 = 2$ ,  $x_2 = 4$ , and  $z = 180$ . (We will see how we find these values later on.)

#### Note 1. [LP Assumptions]

##### 1. The Proportionality and Additivity Assumptions.

The fact that the objective function for an LP must be a linear function of the decision variables has two implications.

- The contribution of the objective function from each decision variable is proportional to the value of the decision variable. For example, the contribution to the objective function in example (1.3) from making 5 desks is exactly five times the contribution to the objective function from making one desk.
- The contribution to the objective function for any variable is independent of the values of the other decision variables. For example, no matter what the value of  $x_2$ , the manufacture of  $x_1$  desks will always contribute  $40x_1$  dollars to the objective function.

Analogously, the fact that each LP constraint must be a linear inequality or linear equation has two implications.

- The contribution of each variable to the lefthand side of each constraint is proportional to the value of the variable.

- (b) The contribution of a variable to the lefthand side of each constraint is independent of the values of the variable.

**2. The Divisibility Assumption.**

The divisibility assumption requires that each decision variable be allowed to assume fractional values. For instance, in example (1.3), the Divisibility Assumption implies that it is acceptable to produce 1.5 desks or 1.63 chairs. Because Frunco cannot actually produce a fractional number of desks or chairs, the Divisibility Assumption is not satisfied in the Frunco problem. A linear programming problem in which some or all of the variables must be nonnegative integers is called an *integer programming problem*.

**3. The Certainty Assumption.**

The certainty assumption is that each parameter (objective function coefficient, righthand side, and constraint coefficient) is known with certainty. If we were unsure of the exact amount of wood used by desks and chairs, the Certainty Assumption would be violated.

### 1.3 Modeling LP Problems

This section presents LP models in which the definition of the variables and the construction of the objective function and constraints are not straightforward.

**Example 1.4.** Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by \$10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Giapetto wants to maximize weekly profit (revenues - costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

**Solution:** Let  $x_1$  = number of soldiers produced each week,  $x_2$  = number of trains produced each week. Then, the formulation of the problem is

$$\begin{array}{ll} \max & z = 3x_1 + 2x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$



**Example 1.5.** A psychologist uses two types of boxes with mice and rats. The amount of time (in minutes) that each mouse and each rat spends in each box per day is given in the table. Write an LP to maximize the number of mice and rats that can be used in this experiment?



	Time		Maximum Time Available per Day
	Mice ( $x$ )	Rats ( $y$ )	
Box A	10 min	20 min	800 min
Box B	20 min	10 min	640 min

**Solution:** Let  $x$  be the number of mice, and  $y$  be the number of rats. Then a correct formulation is

$$\begin{aligned}
 \text{max} \quad & z = x + y \\
 \text{s.t.} \quad & 10x + 20y \leq 800 \\
 & 20x + 10y \leq 640 \\
 & x, y \geq 0
 \end{aligned}$$

**Example 1.6.** A banquet hall offers two types of tables for rent: 6-person rectangular tables at a cost of \$28 each and 10-person round tables at a cost of \$52 each. Kathleen would like to rent the hall for a wedding banquet and needs tables for 250 people. The room can have a maximum of 35 tables and the hall only has 15 rectangular tables available. Formulate an LP to minimize the cost of each type of tables should be rented.

**Solution:** Let  $x_1$  = number of rectangular tables,  $x_2$  = number of round tables. Then, the formulation of the problem is

$$\begin{aligned}
 \text{min} \quad & w = 28x_1 + 52x_2 \\
 \text{s.t.} \quad & 6x_1 + 10x_2 \geq 250 \\
 & x_1 + x_2 \leq 35 \\
 & x_1 \leq 15 \\
 & x_1 \geq 0, x_2 \geq 0
 \end{aligned}$$

### Exercise 1.1.

1. Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs \$4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs \$1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily. Formulate an LP to minimize Leary Chemical's cost of daily demands.
2. A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B. However, the company

cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A and 4 lb per unit of B. The profit units for A and B are \$20 and \$50, respectively. Formulate an LP to maximize the company's profit.

3. A diet is to contain at least 400 units of vitamins, 500 units of minerals, and 1400 calories. Two foods are available: F1, which costs \$0.05 per unit, and F2, which costs \$0.03 per unit. A unit of food F1 contains 2 units of vitamins, 1 unit of minerals, and 4 calories; a unit of food F2 contains 1 unit of vitamins, 2 units of minerals, and 4 calories. Formulate an LP to minimize the cost for a diet that consists of a mixture of these two foods and also meets the minimal nutrition requirements.
4. Katy needs at least 60 units of carbohydrates, 45 units of protein, and 30 units of fat each month. From each pound of food A, she receives 5 units of carbohydrates, 3 of protein, and 4 of fat. Food B contains 2 units of carbohydrates, 2 units of protein, and 1 unit of fat per pound. If food A costs \$1.30 per pound and food B costs \$0.80 per pound, formulate an LP to minimize the cost of pounds of each food should Katy buy each month.
5. An electronics firm manufactures two types of personal computers: a standard model and a portable model. The production of a standard computer requires a capital expenditure of \$400 and 40 hours of labor. The production of a portable computer requires a capital expenditure of \$250 and 30 hours of labor. The firm has \$20000 capital and 2160 labor-hours available for production of standard and portable computers. If each standard computer contributes a profit of \$320 and each portable model contributes a profit of \$220, formulate an LP to maximize the firm's profit.
6. A farmer owns a 100-acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs \$40, \$20, and \$30 per acre, respectively. A maximum of \$3200 can be spent on seed. Crops A, B, and C require one, two, and one work days per acre, respectively, and there are a maximum of 160 work days available. If the farmer can make a profit of \$100 per acre on crop A, \$300 per acre on crop B, and \$200 per acre on crop C, how many acres of each crop should be planted to maximize profit?



7. Farmer Jones bakes two types of cake (chocolate and vanilla) to supplement his income. Each chocolate cake can be sold for \$1, and each vanilla cake can be sold for 0.5\$. Each chocolate cake requires 20 minutes of baking time and uses 4 eggs. Each vanilla cake requires 40 minutes of baking time and uses 1 egg. Eight hours of baking time and 30 eggs are available. Let  $x_1$  be the number of chocolate cakes baked, and  $x_2$  be the number of vanilla cakes baked. Formulate an LP to maximize Farmer Jones's revenue.





## 1.4 Geometric Preliminaries and Solutions

Any LP with only two variables can be solved graphically. We always label the variables  $x_1$  and  $x_2$  and the coordinate axes the  $x_1$  and  $x_2$  axes.

### 1.4.1 Half-Spaces, Hyperplanes, and Convex Sets

**Definition 1.4.** We define the *Euclidean plane*  $\mathbb{R}^n$  to be the set of all  $n$ -tuples of real numbers; that is

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n \}$$

For example,  $\mathbb{R}^2 = \{ (x_1, x_2) \mid x_1 \text{ and } x_2 \text{ are reals} \}$ . Geometrically, we represent  $\mathbb{R}^2$  as in Figure 1.

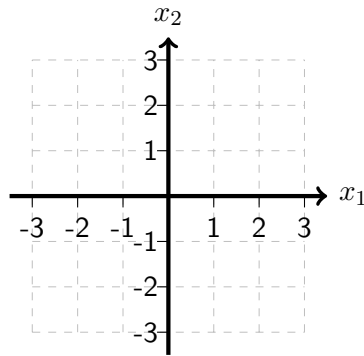


Figure 1 Coordinates Plane,  $\mathbb{R}^2$

The graph in  $\mathbb{R}^2$  of an equation of the form  $a_1x_1 + a_2x_2 = c$  (where  $a_1, a_2, c$  are constants) is a straight line. For example, the graph in  $\mathbb{R}^2$  of the equation  $2x_1 - 3x_2 = 6$  is the line indicated in Figure 2.

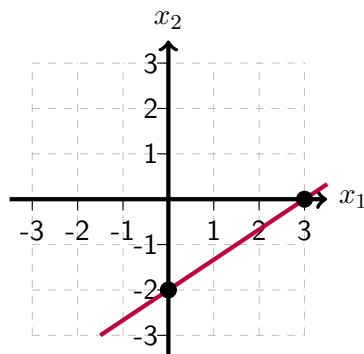


Figure 2 The line  $2x_1 - 3x_2 = 6$

The graph in  $\mathbb{R}^2$  of the inequality  $a_1x_1 + a_2x_2 \leq c$  or  $a_1x_1 + a_2x_2 \geq c$  is the set of all points in  $\mathbb{R}^2$  lying on the line  $a_1x_1 + a_2x_2 = c$  together with all points lying to one side of this line. For example, the shaded region in Figure 3 is the graph of the inequality  $2x_1 - 3x_2 \leq 6$ . To determine on which side of

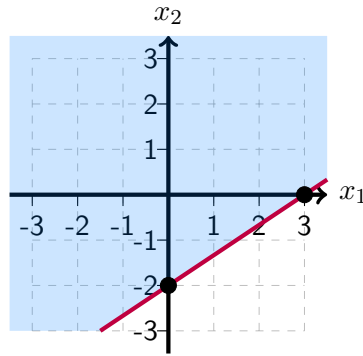


Figure 3 The inequality  $2x_1 - 3x_2 \leq 6$

the line where the region of the inequality  $2x_1 - 3x_2 \leq 6$  lies, consider a point, say  $(0, 0)$ , not lying on the line but satisfying the inequality; the side of the line containing this point is the one corresponding to the inequality.

**Definition 1.5.** A *half-space* in  $\mathbb{R}^n$  is the set of all points in  $\mathbb{R}^n$  satisfying an inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c$$

or an inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$$

where at least one of the constants  $a_1, a_2, \dots, a_n$  is nonzero.

**Definition 1.6.** A *hyperplane* in  $\mathbb{R}^n$  is the set of all points in  $\mathbb{R}^n$  satisfying an equality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where at least one of the constants  $a_1, a_2, \dots, a_n$  is nonzero.

For example, the set of points in  $\mathbb{R}^5$  satisfying

$$3x_1 + \frac{1}{2}x_2 - x_3 + x_4 + \frac{2}{3}x_5 = -9$$

is a hyperplane in  $\mathbb{R}^5$ , and the set of points in  $\mathbb{R}^5$  satisfying

$$3x_1 + \frac{1}{2}x_2 - x_3 + x_4 + \frac{2}{3}x_5 \geq -9$$

is a half-space in  $\mathbb{R}^5$ .

**Definition 1.7.** A subset  $K$  of  $\mathbb{R}^n$  is *convex* if  $K$  is empty, or  $K$  is a single point, or if for each two distinct points  $\mathbf{p}$  and  $\mathbf{q}$  in  $K$ , the line segment connecting  $\mathbf{p}$  and  $\mathbf{q}$  lies entirely in  $K$ .

**Example 1.7.** The sets in Figure 4 are convex.

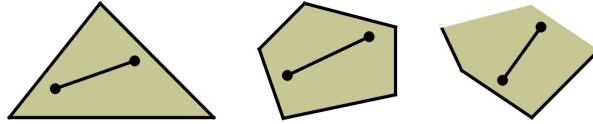


Figure 4 Convex sets

**Example 1.8.** The sets in Figure 5 are not convex.

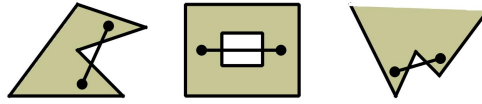


Figure 5 Not convex sets

**Definition 1.8.** If  $\mathbf{p} = (p_1, p_2, \dots, p_n)$  and  $\mathbf{q} = (q_1, q_2, \dots, q_n)$  are points in  $\mathbb{R}^n$ , then the *line segment* joining  $\mathbf{p}$  and  $\mathbf{q}$  consists of all points of the form

$$(1 - t)\mathbf{p} + t\mathbf{q}; \quad 0 \leq t \leq 1$$

where

$$\begin{aligned} (1 - t)\mathbf{p} + t\mathbf{q} &= (1 - t)(p_1, p_2, \dots, p_n) + t(q_1, q_2, \dots, q_n) \\ &= [(1 - t)p_1 + tq_1, \dots, (1 - t)p_n + tq_n] \end{aligned}$$

Observe that if  $t = 0$ , then  $(1 - t)\mathbf{p} + t\mathbf{q} = \mathbf{p}$ , and if  $t = 1$ , then  $(1 - t)\mathbf{p} + t\mathbf{q} = \mathbf{q}$ .

**Example 1.9.** The line segment in  $\mathbb{R}^2$  joining the points  $\mathbf{p} = (3, 6)$  and  $\mathbf{q} = (-4, 5)$  is the set of points

$$\begin{aligned}(1-t)\mathbf{p} + t\mathbf{q} &= (1-t)(3, 6) + t(-4, 5) \\ &= [3(1-t) - 4t, 6(1-t) + 5t] = (3 - 7t, 6 - t) ; 0 \leq t \leq 1\end{aligned}$$

**Theorem 1.1.** A half-space  $H$  in  $\mathbb{R}^n$  that is defined either by the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c$$

or the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$$

is convex.

*Proof.* We establish this result for the half-space  $H$  defined by the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c \quad (1.1)$$

A similar argument hold for half-spaces defined by  $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$ . Suppose the points  $\mathbf{p} = (p_1, \cdots, p_n)$  and  $\mathbf{q} = (q_1, \cdots, q_n)$  lie in  $H$ ; that is, these points satisfy inequality (1.1), so we have

$$a_1p_1 + a_2p_2 + \cdots + a_np_n \leq c$$

$$a_1q_1 + a_2q_2 + \cdots + a_nq_n \leq c$$

To show that the line segment connecting these two points lies entirely in  $H$ , it suffices to show that for each  $t \in [0, 1]$ , the point

$$(1-t)\mathbf{p} + t\mathbf{q} = [(1-t)p_1 + tq_1, \cdots, (1-t)p_n + tq_n]$$

also satisfies inequality (1.1). To show this, we have

$$\begin{aligned}&a_1[(1-t)p_1 + tq_1] + a_2[(1-t)p_2 + tq_2] + \cdots + a_n[(1-t)p_n + tq_n] \\ &= (1-t)(a_1p_1 + a_2p_2 + \cdots + a_np_n) + t(a_1q_1 + a_2q_2 + \cdots + a_nq_n) \\ &\leq (1-t)c + tc = c\end{aligned}$$

and this concludes the proof. ■

**Theorem 1.2.** If  $K_1, K_2, \dots, K_r$  are convex subsets of  $\mathbb{R}^n$ , then the intersection of these sets,  $K = K_1 \cap K_2 \cap \dots \cap K_r$  is also convex.

*Proof.* If  $K$  is empty or consists of a single point, then  $K$  is convex by definition (1.7). Suppose then that  $K$  consists of more than one point, and let  $\mathbf{p}$  and  $\mathbf{q}$  be any two distinct points in  $K$ . Since  $\mathbf{p}$  and  $\mathbf{q}$  are in each convex set  $K_i$ ;  $1 \leq i \leq r$ , the line segment  $L$  connecting  $\mathbf{p}$  and  $\mathbf{q}$  also lies entirely in each  $K_i$ . Therefore,  $L$  lies in the intersection  $K$  of these sets, and we conclude that  $K$  is convex. ■

**Theorem 1.3.** A hyperplane  $M$  in  $\mathbb{R}^n$  defined by

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = c$$

is convex

*Proof.*  $M$  is the intersection of the convex half-spaces

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq c$$

and

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \geq c$$

By Theorem (1.2), this intersection is convex. ■

**Definition 1.9.** A point  $\mathbf{q}$  is a *corner point* (or an *extreme point*) of a convex set  $K$  if  $\mathbf{q}$  is not an interior point of any line segment contained in  $K$ .

**Example 1.10.** The points  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5$  are corner points of the convex set in Figure 6.

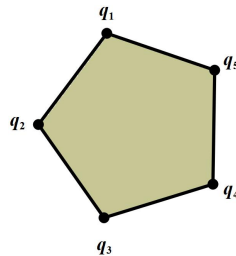


Figure 6

### Exercise 1.2.

1. Draw the graph in  $\mathbb{R}^2$  of the following half-spaces.

- (a)  $-2x_1 + 4x_2 \geq 12$ .                      (c)  $x_1 \geq 4$ .  
 (b)  $x_2 \leq 2x_1$ .                                (d)  $-3x_2 \leq 9$

2. Which of the following expressions define hyperplanes, half-spaces, or neither?

- (a)  $2x_1 + 3x_2 = x_2 - x_4 + 3$ .                      (d)  $x_1 = 6 + \frac{2}{x_2}$ .  
 (b)  $x_1 - 3x_4 \geq 3x_2 + x_3$ .                      (e)  $2.5x_1 - 3.2x_2 = 10$ .  
 (c)  $x_1x_2 \leq 1$ .                                        (f)  $x_1 + x_2^3 \geq 9$ .

3. Which of the following sets in Figure 7 are convex?

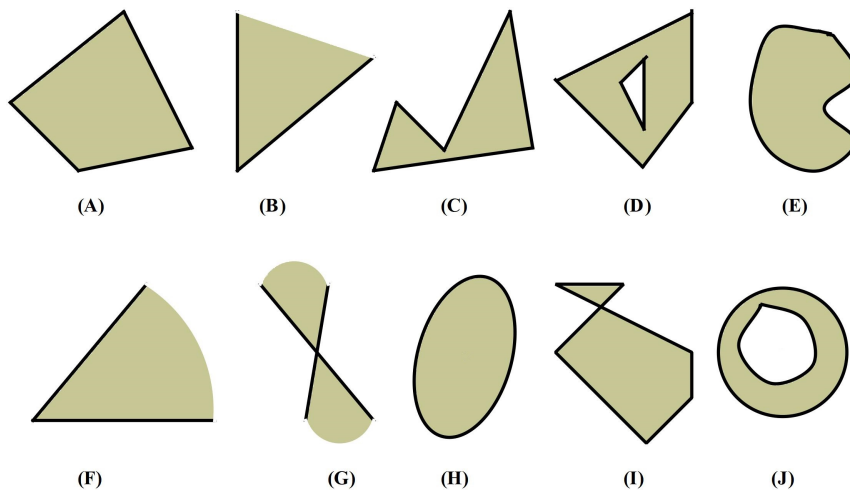


Figure 7

4. Let  $\mathbf{p} = (1, 3, 2)$  and  $\mathbf{q} = (2, 4, -1)$  be two points in  $\mathbb{R}^3$ .

- (a) Find the set of points that lie on the line segment joining the points  $\mathbf{p}$  and  $\mathbf{q}$ .  
 (b) Show that the point  $(1.5, 3.5, 0.5)$  lies on the line segment joining the points  $\mathbf{p}$  and  $\mathbf{q}$ .

### 1.4.2 The Graphical Solution of Two–Variable LP Problems

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution. For defining these concepts, we use the term *point* to mean a specification of the value for each decision variable.

**Definition 1.10.** The *feasible region* for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions. Any point that is not in LP's feasible region is said to be *infeasible point*.

The shaded area in Figure 8 indicates the feasible region of the LP in example (1.3). Note that each of the constraints in the LP defines a half-space. The feasible set consists of all points in the intersection of these half-spaces. Observe that the feasible region in Figure 8 is convex. Note that the points  $(0, 0)$ ,  $(1, 3)$ ,

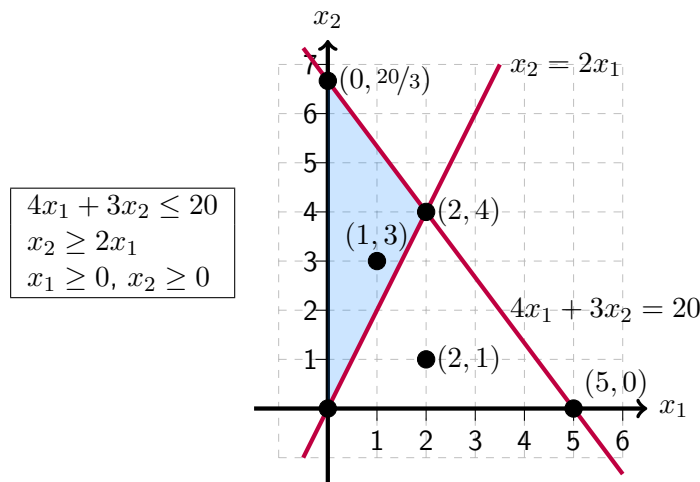


Figure 8 The feasible region of example (1.3)

and  $(2, 4)$  are all in the feasible region, while  $(2, 1)$  is infeasible, because it does not satisfy the second constraint.

**Theorem 1.4.** The feasible region in  $\mathbb{R}^n$  corresponding to any number of constraints of the types

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

$$x_1, x_2, \cdots, x_n \geq 0$$

is convex.

*Proof.* The inequality constraints define half-spaces, and the equality constraints define hyperplanes. By Theorems 1.1 and 1.3 these half-spaces and hyperplanes are convex sets. Since the feasible region is the intersection of these convex sets, it follows from Theorem 1.2 that the feasible region is convex. ■

**Definition 1.11.** For a maximization (minimization) problem, an *optimal solution* to an LP is a point in the feasible region with the largest (smallest) objective function value.

The goal of any LP problem is to find the optimum, the best feasible solution that maximizes the total profit or minimizes the cost. Having identified the feasible region for the Furnco problem in example (1.3) as shown in Figure 8, we now search for the optimal solution, which will be the point in the feasible region with the largest value of  $z = 40x_1 + 25x_2$ .

- To find the optimal solution, we need to graph a line on which all points have the same  $z$ -value. In a max problem, such a line is called an *isoprofit* line.
- To draw an isoprofit line, choose any point in the feasible region and calculate its  $z$ -value. Let us choose  $(1, 3)$ . For  $(1, 3)$ ,  $z = 40(1) + 25(3) = 115$ . Thus,  $(1, 3)$  lies on the isoprofit line  $z = 40x_1 + 25x_2 = 115$ .
- Because all isoprofit lines are of the form  $40x_1 + 25x_2 = \text{constant}$ , all isoprofit lines have the same slope. This means that once we have drawn one isoprofit line, we can find all other isoprofit lines by moving parallel to the isoprofit line we have drawn in a direction that increases  $z$ .
- After a point, the isoprofit lines will no longer intersect the feasible region. The last isoprofit line intersecting (touching) the feasible region defines the largest  $z$ -value of any point in the feasible region and indicates the optimal solution to the LP.
- In our problem, the objective function  $z = 40x_1 + 25x_2$  will increase if we move in a direction for which both  $x_1$  and  $x_2$  increase. Thus, we construct additional isoprofit lines by moving parallel to  $40x_1 + 25x_2 = 115$  in a northeast direction (upward and to the right), as shown in Figure 9.
- From Figure 9, we see that the isoprofit line passing through point  $(2, 4)$  is the last isoprofit line to intersect the feasible region. Thus,  $(2, 4)$  is the point in the feasible region with the largest  $z$ -value and is therefore the optimal solution to the Furnco problem. Thus, the optimal value of  $z$  is  $z = 40(2) + 25(4) = 180$ .

**Note 2.** In a minimization problem, to find the optimal solution, we need to graph a line on which all points have the same  $w$ -value, such a line is called an *isocost* line. Once we have drawn one isocost line, we can find all other isocost lines by moving parallel to the isocost line we have drawn first in a direction that decreases  $w$ . After a point, the isocost lines will no longer intersect the



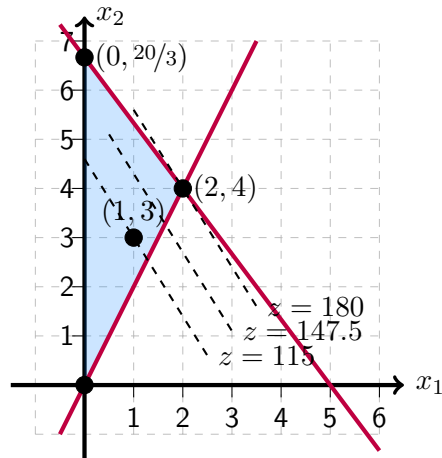


Figure 9 Isoprofit lines

feasible region. The last isocost line intersecting the feasible region defines the smallest  $w$ -value of any point in the feasible region and indicates the optimal solution to the LP.

**Example 1.11.** Graphically solve the following LP problem.

$$\begin{aligned}
 \min \quad & w = -4x_1 + 7x_2 \\
 \text{s.t.} \quad & x_1 + x_2 \geq 3 \\
 & -x_1 + x_2 \leq 3 \\
 & 2x_1 + x_2 \leq 8 \\
 & x_1 \geq 0, x_2 \geq 0
 \end{aligned}$$

**Solution:** From Figure 10, The isocost lines minimize the value of the objective function by decreasing  $w$  in the southeast direction (down-right). The optimum solution is the intersection of the two lines  $2x_1 + x_2 = 8$  and  $x_2 = 0$ , which yields  $x_1 = 4$  and  $x_2 = 0$ . The minimum value of  $w$  is  $w = -4(4) + 7(0) = -16$ .

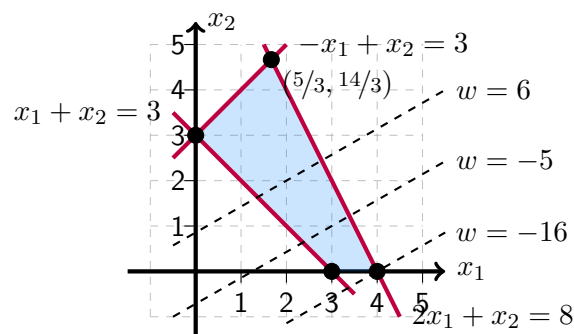


Figure 10

**The Corner-Point Theorem** In practice, a typical LP may include hundreds or even thousands of variables and constraints. Of what good then is the study of a two-variable LP? The answer is that the graphical solution provides one of the most important key result in linear programming: “*The optimum solution of an LP, when it exists, is always associated with a corner point of the solution space*”, thus limiting the search for the optimum from an infinite number of feasible points to a finite number of corner points. This powerful result is the basis for the development of the general algebraic *simplex method* presented in Chapter 2.

**Theorem 1.5. [Corner Point Theorem]**

*Consider the LP problem*

$$\text{Maximize ( or Minimize) } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

*subject to a system of inequalities of the types*

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \leq b$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \geq b$$

$$x_1, x_2, \cdots, x_n \geq 0$$

1. *If the feasible region is bounded, then the optimal solution is attained at a corner point of this feasible region.*
2. *If the feasible region is unbounded, then an optimal solution may not exist; however, if an optimal solution exists, it is attained at a corner point of this feasible region.*

*Proof.* The proof of this theorem can be found in: **Jan Van Tiel, Convex Analysis, New York: Wiley, 1984.** ■

**Note 3.** If the feasible region of an LP is bounded, then an optimal solution exists and we use The Corner Point Theorem (1.5) to find this optimal solution. In the case that the feasible region is unbounded, the isoprofit (isocost) line is used to determine whether the LP has an optimal solution or not.

**Example 1.12.** Graphically solve the following LP problem.

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

**Solution:** From Figure 11, the feasible region is bounded, so we use Theorem (1.5) as follows:

Corner	$z$
$(0, 0)$	0
$(0, 80)$	160
$(20, 60)$	180
$(40, 20)$	160
$(40, 0)$	120

**Optimal**

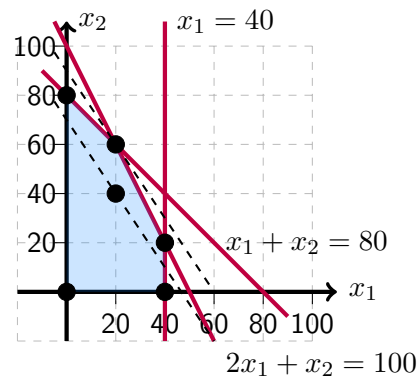


Figure 11

Once the optimal solution to an LP has been found, it is useful to classify each constraint as being a binding constraint or a nonbinding constraint.

**Definition 1.12.** A constraint is *binding* if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint. Otherwise, it is called *nonbinding*.

For instance, in example (1.12), the first two constraints are binding, while the third one is nonbinding. While in example (1.11) the third constraint is binding and the other two constraints are nonbinding.

**Definition 1.13.** A constraint is said to be *redundant* if its removal from the model leaves the feasible solution space unchanged.

For example, the feasible region of the following constraints has a redundant constraint as shown in Figure 12

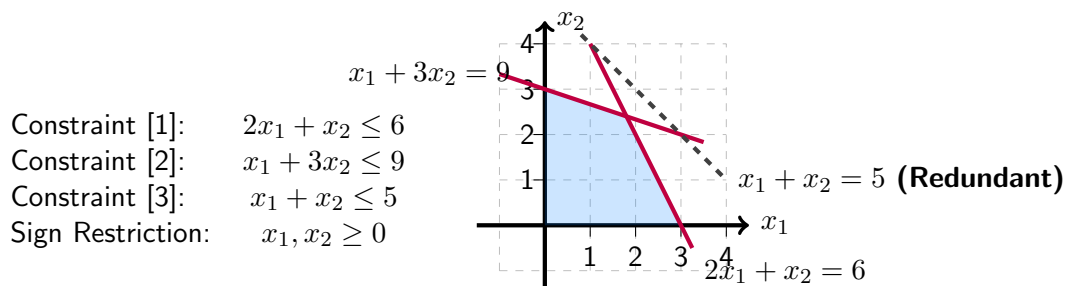


Figure 12

Note that the third constraint is the redundant constraint since its removal from the region will leave the feasible region unchanged.

**Note 4. [Alternative Optima]** An LP problem may have an infinite number of alternative optima when the objective function is parallel to a nonredundant binding constraint. This is indicated by the fact that as an isoprofit (or isocost) line leaves the feasible region, it will intersect an entire line segment corresponding to the binding constraint. It seems reasonable that if two points are optimal, then any point on the line segment joining these two points will also be optimal. If an alternative optimum occurs, then the decision maker can use a secondary criterion to choose between optimal solutions. The technique of *goal programming* is often used to choose among alternative optimal solutions. The next example demonstrates the practical significance of such solutions.

**Example 1.13.** Graphically solve the following LP problem.

$$\begin{array}{ll}\max & z = 4x_1 + x_2 \\ \text{s.t.} & 8x_1 + 2x_2 \leq 16 \\ & 5x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** The feasible region for this LP is the shaded bounded region in Figure 13. So we use Theorem (1.5) as follows:

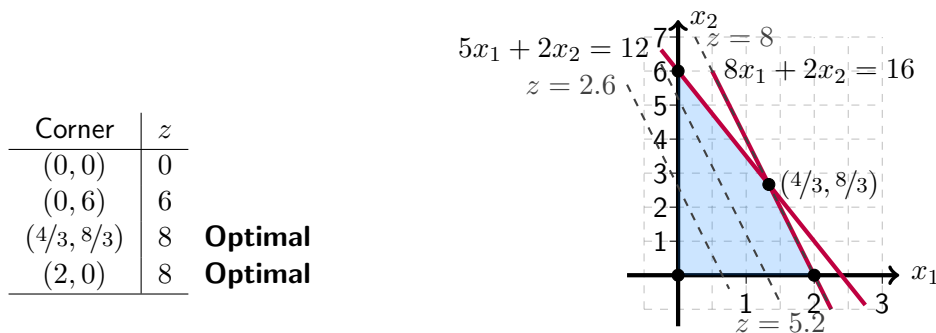


Figure 13

From the table above, the optimal solution occurs at the two corner points  $(\frac{4}{3}, \frac{8}{3})$  and  $(2, 0)$ . This means that any point on line segment joining these corner points optimal. These points are given by

$$t \left( \frac{4}{3}, \frac{8}{3} \right) + (1 - t)(2, 0) = \left( 2 - \frac{2t}{3}, \frac{8t}{3} \right) ; 0 \leq t \leq 1$$

**Note 5. [Infeasible LP]** It is possible for an LP's feasible region to be empty (contain no points), resulting in an infeasible LP. Because the optimal solution to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.

**Example 1.14.** The following LP problem has no feasible solution as shown in Figure 14.

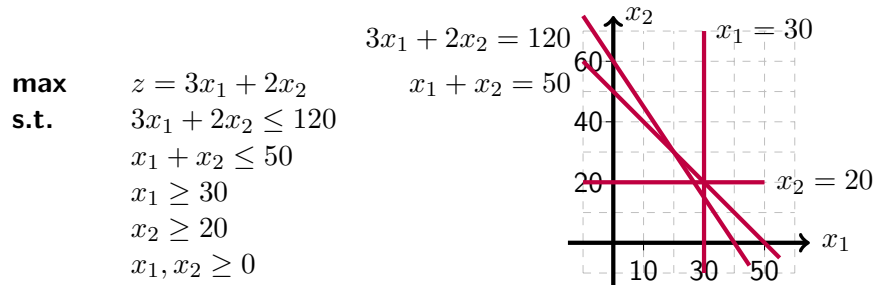


Figure 14

**Note 6. [Unbounded LP]** For a max problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large  $z$ -values, which corresponds to a decision maker earning arbitrarily large revenues or profits. This would indicate that an unbounded optimal solution should not occur in a correctly formulated LP. Thus, if the reader ever solves an LP on the computer and finds that the LP is unbounded, then an error has probably been made in formulating the LP or in inputting the LP into the computer. For a minimization problem, an LP is unbounded if there are points in the feasible region with arbitrarily small  $z$ -values. When graphically solving an LP, we can spot an unbounded LP as follows:

- A max problem is unbounded if, when we move parallel to our original isoprofit line in the direction of increasing  $z$ , we never entirely leave the feasible region.
- A minimization problem is unbounded if we never leave the feasible region when moving in the direction of decreasing  $z$ .

**Example 1.15.** Graphically solve the following LP problem.

$$\begin{array}{ll} \max & z = 2x_1 - x_2 \\ \text{s.t.} & x_1 - x_2 \leq 1 \\ & 2x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

**Solution:** The feasible region is the (shaded) unbounded region in Figure 15. To find the optimal solution, we draw the isoprofit line passing through  $(2, 5)$ . This isoprofit line has  $z = 2(2) - (5) = -1$ . The direction of increasing  $z$  is to the southeast (this makes  $x_1$  larger and  $x_2$  smaller). Moving parallel to  $z = 2x_1 - x_2$  in a southeast direction, we see that any isoprofit line we draw will intersect the feasible region. (This is because any isoprofit line is steeper than the line  $x_1 - x_2 = 1$ .) Thus, there are points in the feasible region that have arbitrarily large  $z$ -values.

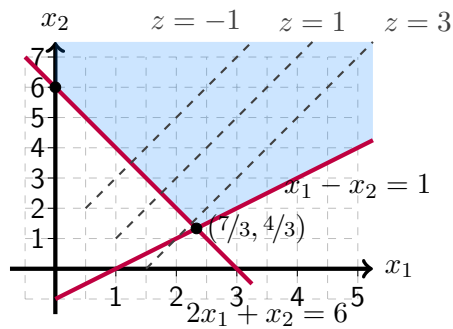


Figure 15

**Example 1.16.** Graphically solve the following LP problem.

$$\begin{array}{ll}
 \min & w = x_1 + 2x_2 \\
 \text{s.t.} & x_1 + x_2 \geq 6 \\
 & 2x_1 - x_2 \geq 0 \\
 & x_1 - 2x_2 \leq 0 \\
 & x_1, x_2 \geq 0
 \end{array}$$

**Solution:** Although the feasible region of the LP is unbounded, but it has optimal solution as shown in Figure 16.

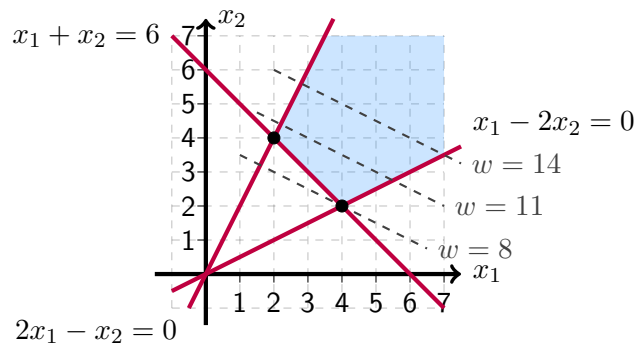


Figure 16

**Example 1.17.** Graphically solve the following LP problem.

$$\begin{array}{ll}
 \max & z = 5x_1 + 6x_2 \\
 \text{s.t.} & x_1 - 2x_2 \geq 1 \\
 & -2x_1 + x_2 \geq 1 \\
 & x_1, x_2 \text{ urs}
 \end{array}$$

**Solution:** The two variables  $x_1$  and  $x_2$  are unrestricted in sign means that both can be positive, negative, or zero. The feasible region is the shaded region

in Figure 17. To find the optimal solution, we draw the isoprofit line passing through  $(-3, 3)$ . This isoprofit line has  $z = 5(-3) + 6(-3) = -33$ . The direction of increasing  $z$  is to the northeast. Moving parallel to  $z = 5x_1 + 6x_2$  in a northeast direction, we see that the last isoprofit line we draw will touch the feasible region at the point  $(-1, -1)$ . Thus, the optimal  $z$ -value is  $z = 5(-1) + 6(-1) = -11$ .

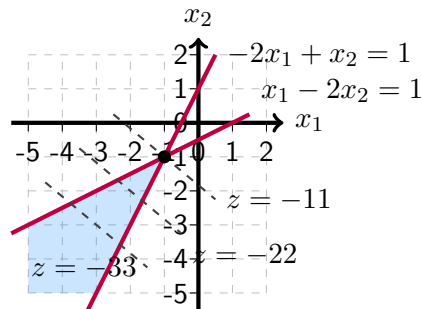


Figure 17

### Exercise 1.3.

- Match the solution region of each system of linear inequalities with one of the four regions shown in Figure 18.

(a)  $x + 2y \leq 8$   
 $3x - 2y \geq 0$

(b)  $x + 2y \geq 8$   
 $3x - 2y \leq 0$

(c)  $x + 2y \geq 8$   
 $3x - 2y \geq 0$

(d)  $x + 2y \leq 8$   
 $3x - 2y \leq 0$

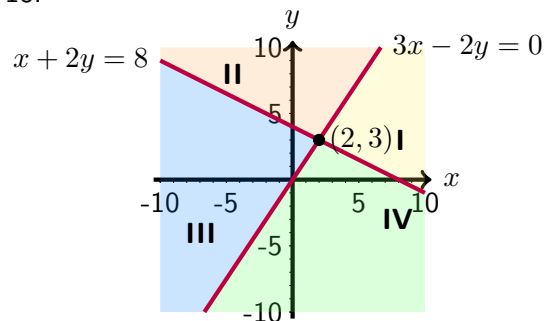


Figure 18

- Find the maximum value of each objective function over the feasible region shown in Figure 19.

(a)  $z = x + y$

(b)  $z = 4x + y$

(c)  $z = 3x + 7y$

(d)  $z = 9x + 3y$

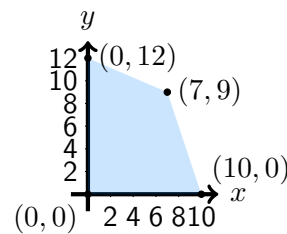


Figure 19

3. Find the minimum value of each objective function over the feasible region shown in Figure 20.

(a)  $w = 7x + 4y$

(b)  $w = 7x + 9y$

(c)  $w = 3x + 8y$

(d)  $w = 5x + 4y$

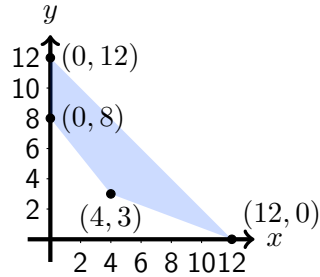


Figure 20

4. The corner points for the bounded feasible region determined by the system of linear inequalities

$$x + 2y \leq 10$$

$$3x + y \leq 15$$

$$x, y \geq 0$$

are  $O = (0,0)$ ,  $A = (0,5)$ ,  $B = (4,3)$ , and  $C = (5,0)$  as shown in Figure 21. If  $P = ax + by$  and  $a, b > 0$ , determine conditions on  $a$  and  $b$  that will ensure that the maximum value of  $P$  occurs

(a) only at  $A$

(b) only at  $B$

(c) only at  $C$

(d) at both  $A$  and  $B$

(e) at both  $B$  and  $C$

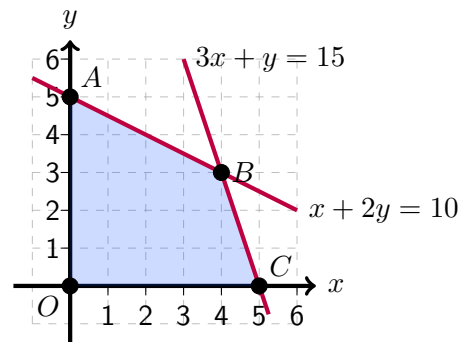


Figure 21

5. Identify the direction of increase in  $z$  in each of the following cases:

(a) Maximize  $z = x_1 - x_2$ .

(b) Maximize  $z = -8x_1 - 3x_2$ .

(c) Maximize  $z = -x_1 + 3x_2$ .



6. Identify the direction of decrease in  $w$  in each of the following cases:

(a) Minimize  $w = 4x_1 - 2x_2$ .

(b) Minimize  $w = -6x_1 + 2x_2$ .

7. Determine the solution space graphically for the following inequalities. Which constraints are redundant? Reduce the system to the smallest number of constraints that will define the same solution space.

$$\begin{aligned} x + y &\leq 4 \\ 4x + 3y &\leq 12 \\ -x + y &\geq 1 \\ x + y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

8. Write the constraints associated with the solution space shown in Figure 22 and identify the redundant constraints.

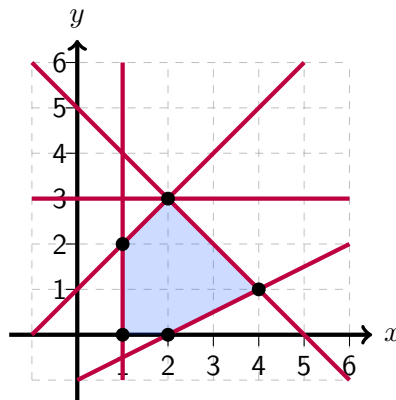


Figure 22

9. Consider the following problem:

$$\begin{aligned} \max \quad & z = 6x_1 - 2x_2 \\ \text{s.t.} \quad & 3x_1 - x_2 \leq 6 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Show graphically that at the optimal solution, the variables  $x_1$  and  $x_2$  can be increased indefinitely while the value of the objective function remains constant.

10. Consider the following problem:

$$\begin{array}{ll}\mathbf{max} & z = 3x_1 + 2x_2 \\ \mathbf{s.t.} & 2x_1 + x_2 \leq 2 \\ & 3x_1 + 4x_2 \geq 12 \\ & x_1, x_2 \geq 0\end{array}$$

Show graphically that the problem has no feasible optimal solution.

11. Solve the following problem by *inspection* without graphing the feasible region.

$$\begin{array}{ll}\mathbf{min} & w = 5x_1 + 2x_2 \\ \mathbf{s.t.} & x_1 + x_2 = 5 \\ & 7x_1 - 5x_2 = -1 \\ & x_1, x_2 \geq 0\end{array}$$

12. Solve the following problems graphically.

$$\begin{array}{ll}\text{(a)} \quad \mathbf{min} & w = 4x_1 + x_2 \\ \mathbf{s.t.} & 3x_1 + x_2 \geq 6 \\ & 4x_1 + x_2 \geq 12 \\ & x_1 \geq 2 \\ & x_1, x_2 \geq 0\end{array}$$

$$\begin{array}{ll}\text{(b)} \quad \mathbf{max} & z = 5x_1 - x_2 \\ \mathbf{s.t.} & 2x_1 + 3x_2 \geq 12 \\ & x_1 - 3x_2 \geq 0 \\ & x_1, x_2 \geq 0\end{array}$$

$$\begin{array}{ll}\text{(c)} \quad \mathbf{min} & w = 5x_1 + x_2 \\ \mathbf{s.t.} & 2x_1 + x_2 \geq 6 \\ & x_1 + x_2 \geq 4 \\ & x_1 + 5x_2 \geq 10 \\ & x_1, x_2 \geq 0\end{array}$$

$$\begin{array}{ll}\text{(d)} \quad \mathbf{max} & z = -4x_1 + x_2 \\ \mathbf{s.t.} & 3x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 12 \\ & x_1 \geq 0 \text{ and } x_2 \text{ urs}\end{array}$$

$$\begin{array}{ll}\text{(e)} \quad \mathbf{max} & z = x_1 + 5x_2 \\ \mathbf{s.t.} & x_1 - 3x_2 \geq 0 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0\end{array}$$

$$\begin{array}{ll}\text{(f)} \quad \mathbf{min} & w = 4x_1 + x_2 \\ \mathbf{s.t.} & 3x_1 + x_2 \geq 10 \\ & x_1 + x_2 \geq 5 \\ & x_1 \geq 3 \\ & x_1, x_2 \geq 0\end{array}$$

## 2 The Simplex Method

Thus far we have used a geometric approach to solve certain LP problems. We have observed, in Chapter 1, that this procedure is limited to problems of two or three variables. The *simplex algorithm* is essentially algebraic in nature and is more efficient than its geometric counterpart.

### 2.1 The Idea of the Simplex Method

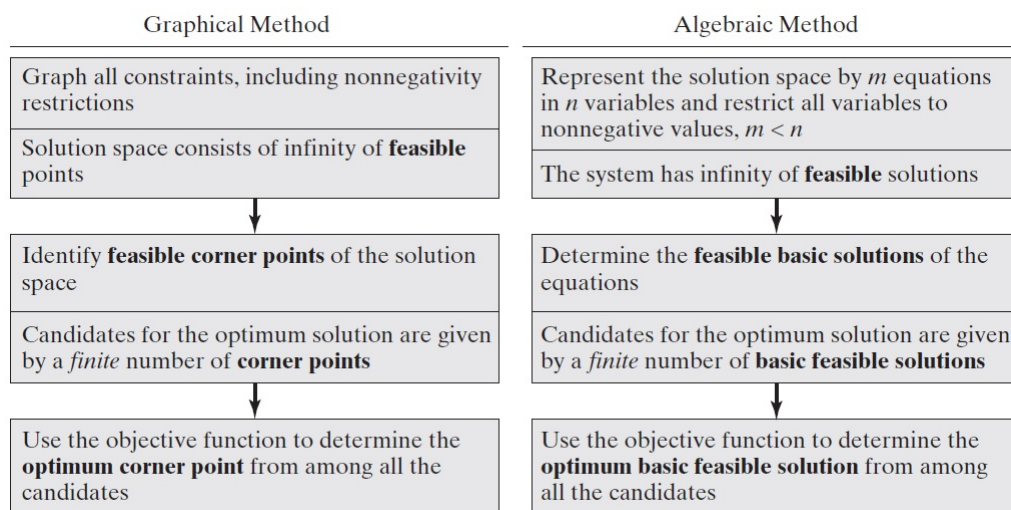
The graphical method represented in Chapter 1 demonstrates that the optimum LP is always associated with a corner point of the solution space. What the simplex method does is to translate the geometric definition of the extreme point into an algebraic definition.

As an initial step, the simplex method requires that each of the constraints be put in a special standard form in which all the constraints are expressed as equations. This conversion results in a set of equations in which the number of variables exceeds the number of equations, which means that the equations yields an infinite number of solution points.

The extreme points of this space can be identified algebraically by setting certain number of variables to zero and then solving for the remaining variables (called the *basic solutions*), provided that the condition results in a unique solution.

Corner Points  $\iff$  Basic Solutions

What the simplex method does is to identify a starting basic solution and then move systematically to other basic solutions that have the potential to improve the value of the objective function. Eventually, the basic solution corresponding to the optimum will be identified and the computational process will end.



## 2.2 Converting an LP to Standard Form

We have seen that an LP can have both equality and inequality constraints. It also can have variables that are required to be nonnegative as well as those allowed to be unrestricted in sign (urs). The development of the simplex method computations is facilitated by imposing two requirements on the LP model:

1. All the constraints are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

An LP in this form is said to be in **standard form**. To convert an LP into standard form, we do the following steps.

1. If the right-hand of a constraint is negative, multiply both sides of the constraint by  $-1$ . This multiplication will convert a  $\leq$  sign to  $\geq$  and vice versa.
2. To convert a  $\leq$  inequality to an equation, a nonnegative **slack** variable (unused amount) is added to the left-hand side of the constraint. For example, the constraint  $3x_1 + 2x_2 \leq 12$  is converted into an equation as

$$3x_1 + 2x_2 + s = 12 ; \quad s \geq 0.$$

3. Conversion from  $\geq$  to  $=$  is achieved by subtracting a nonnegative **surplus** (excess amount) variable from the left-hand side of the inequality. For example, the surplus variable  $e$  converts the constraint  $5x_1 + 3x_2 \geq 4$  to the equation

$$5x_1 + 3x_2 - e = 4 ; \quad e \geq 0.$$

4. An unrestricted variable  $x_i$  can be presented in terms of two nonnegative variables by using the substitution

$$x_i = y_{i1} - y_{i2} ; \quad y_{i1}, y_{i2} \geq 0$$

The substitution must be effected throughout all the constraints and the objective function. In the optimal LP solution only one of the two variables  $y_{i1}$  and  $y_{i2}$  can assume a positive value, but never both. Thus, when  $y_{i1} > 0$ ,  $y_{i2} = 0$  and vice versa. For example, if  $x_i = 4$  then  $y_{i1} = 4$  and  $y_{i2} = 0$ , and if  $x_i = -4$  then  $y_{i1} = 0$  and  $y_{i2} = 4$ .

**Example 2.1.** Write the following LP problem in standard form.

$$\begin{array}{ll} \text{min} & w = 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 = 10 \\ & -2x_1 + 3x_2 \leq -5 \\ & 7x_1 - 4x_2 \leq 6 \\ & x_1 \text{ urs, } x_2 \geq 0 \end{array}$$

**Solution:** The following changes must be effected.

1. Multiply both sides of the second constraint by  $-1$  to get  $2x_1 - 3x_2 \geq 5$ , then subtract excess variable  $e_2 \geq 0$  from the left-hand side of the constraint.
2. Add a slack variable  $s_3 \geq 0$  to the left-hand side of the third constraint.
3. Substitute  $x_1 = y_{11} - y_{12}$ , where  $y_{11}, y_{12} \geq 0$ , in the objective function and all the other constraints.

Thus we get the standard form as

$$\begin{array}{ll}
 \text{min} & w = 2y_{11} - 2y_{12} + 3x_2 \\
 \text{s.t.} & y_{11} - y_{12} + x_2 = 10 \\
 & 2y_{11} - 2y_{12} - 3x_2 - e_2 = 5 \\
 & 7y_{11} - 7y_{12} - 4x_2 + s_3 = 6 \\
 & y_{11}, y_{12}, x_2, e_2, s_3 \geq 0
 \end{array}$$

### Exercise 2.1.

1. Convert the following LP to the standard form.

$$\begin{array}{ll}
 \text{max} & z = 2x_1 + 3x_2 + 5x_3 \\
 \text{s.t.} & x_1 + x_2 - x_3 \geq -5 \\
 & -6x_1 + 7x_2 - 9x_3 \leq 4 \\
 & x_1 + x_2 + 4x_3 = 10 \\
 & x_1, x_2 \geq 0, x_3 \text{ urs}
 \end{array}$$

2. Consider the inequality

$$22x_1 - 4x_2 \geq -7$$

Show that multiplying both sides of the inequality by  $-1$  and then converting the resulting inequality into an equation is the same as converting it first to an equation and then multiplying both sides by  $-1$ .

3. The substitution  $x = y_1 - y_2$  is used in an LP to replace unrestricted  $x$  by the two nonnegative variables  $y_1$  and  $y_2$ . If  $x$  assumes the respective values  $-6$ ,  $10$ , and  $0$ , determine the associated optimal values of  $y_1$  and  $y_2$  in each case.

## 2.3 Basic Feasible Solutions

Suppose we have converted an LP with  $m$  constraints into standard form. Assuming that the standard form contains  $n$  variables (labeled for convenience  $x_1, x_2, \dots, x_n$ ), where  $n \geq m$ , the standard form for such an LP is

$$\begin{array}{ll}
\text{max (or min)} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& x_1, x_2, \dots, x_n \geq 0
\end{array}$$

If we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

the constraints for the LP may be written as the system of equations  $A\mathbf{x} = \mathbf{b}$ .

**Definition 2.1.** A *basic solution* to  $A\mathbf{x} = \mathbf{b}$  is obtained by setting  $n - m$  variables (the nonbasic variables, or NBV) equal to 0 and solving for the values of the remaining  $m$  variables (the basic variables, or BV), provided the resulting solution is unique.

**Note 7.** The maximum number of corner points is  $C_m^n = \frac{n!}{m!(n-m)!}$ .

Of course, the different choices of nonbasic variables will lead to different basic solutions. To illustrate, we find all the basic solutions to the following system of two equations ( $m = 2$ ) in three variables ( $n = 3$ ):

$$\begin{array}{rcl}
x_1 + x_2 & = & 3 \\
-x_2 + x_3 & = & -1
\end{array}$$

We begin by choosing a set of  $n - m = 3 - 2 = 1$  nonbasic variable, and note that there are  $C_2^3 = 3$  choices of this nonbasic variable.

- If NBV =  $\{x_3\}$ , then BV =  $\{x_1, x_2\}$ . We obtain the values of the basic variables by setting  $x_3 = 0$  and solving

$$\begin{array}{rcl}
x_1 + x_2 & = & 3 \\
-x_2 & = & -1
\end{array}$$

We find that  $x_1 = 2$ ,  $x_2 = 1$ . Thus,  $(x_1, x_2, x_3) = (2, 1, 0)$  is a basic solution to the system.

- If NBV =  $\{x_2\}$ , then BV =  $\{x_1, x_3\}$ . We obtain the values of the basic variables by setting  $x_2 = 0$  and we find that  $x_1 = 3$ ,  $x_3 = -1$ . Thus,  $(x_1, x_2, x_3) = (3, 0, -1)$  is a basic solution to the system.

- If  $\text{NBV} = \{x_1\}$ , then  $\text{BV} = \{x_2, x_3\}$ . We obtain the values of the basic variables by setting  $x_1 = 0$  and solving

$$\begin{aligned} x_2 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

We find that  $x_2 = 3$ ,  $x_3 = 2$ . Thus,  $(x_1, x_2, x_3) = (0, 3, 2)$  is a basic solution to the system.

The following table provides all the basic and nonbasic solutions of the above linear system.

NBVs	BVs	Basic Solution
$x_1$	$x_2, x_3$	3, 2
$x_2$	$x_1, x_3$	3, -1
$x_3$	$x_1, x_2$	2, 1

**Note 8.** Some sets of  $m$  variables do not yield a basic solution. For example, consider the following linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + x_3 &= 3 \end{aligned}$$

If we choose  $\text{NBV} = \{x_3\}$  and  $\text{BV} = \{x_1, x_2\}$ , the corresponding basic solution would be obtained by solving

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 3 \end{aligned}$$

Because this system has no solution, there is no basic solution corresponding to  $\text{BV} = \{x_1, x_2\}$ .

**Definition 2.2.** Any basic solution to the constraints  $A\mathbf{x} = \mathbf{b}$  of an LP in which all variables are nonnegative is a *basic feasible solution* (or **bfs**).

For example, for an LP with the constraints given by

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

the basic solutions  $x_1 = 2, x_2 = 1, x_3 = 0$ , and  $x_1 = 0, x_2 = 3, x_3 = 2$  are basic feasible solutions, but the basic solution  $x_1 = 3, x_2 = 0, x_3 = -1$  fails to be a feasible solution (because  $x_3 < 0$ ).

**Theorem 2.1.** *A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.*

**Example 2.2.** Consider the following LP with two variables.

$$\begin{array}{ll} \max & z = 2x_1 + 3x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

**Solution:** By adding slack variables  $s_1$  and  $s_2$ , respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \max & z = 2x_1 + 3x_2 \\ \text{s.t.} & 2x_1 + x_2 + s_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

Figure 23 provides the graphical solution space for the problem. Algebraically,

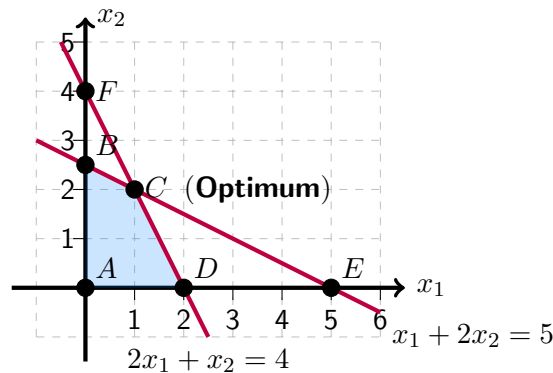


Figure 23

the solution space of the LP is represented by the following  $m = 2$  equations and  $n = 4$  variables:

$$\begin{array}{rcl} 2x_1 + x_2 + s_1 & = & 4 \\ x_1 + 2x_2 + s_2 & = & 5 \end{array}$$

The basic solutions are determined by setting  $n - m = 2$  variables equal to zero and solving for the remaining  $m = 2$  variables. For example, if we set  $x_1 = 0$  and  $x_2 = 0$ , the equations provide the unique basic solution

$$s_1 = 4 \text{ and } s_2 = 5.$$



This solution corresponds to point  $A$  in Figure 23. Another point can be determined by setting  $s_1 = 0$  and  $s_2 = 0$  and then solving the resulting two equations

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

The associated basic solution is  $x_1 = 1$ ,  $x_2 = 2$ , or point  $C$  in Figure 23. In the present example, the maximum number of corner points is  $C_2^4 = 6$ . Looking at Figure 23, we can spot the four corner points  $A$ ,  $B$ ,  $C$ , and  $D$ . So, where are the remaining two? In fact, points  $E$  and  $F$  also are corner points. But, they are infeasible, and, hence, are not candidates for the optimum. The following table provides all the basic and nonbasic solutions of the current example.

NBVs	BVs	Basic Solution	Corner Point	Feasible?	$z$ -Value	
$x_1, x_2$	$s_1, s_2$	4, 5	$A$	✓	0	
$x_1, s_1$	$x_2, s_2$	4, -3	$F$	✗		
$x_1, s_2$	$x_2, s_1$	2.5, 1.5	$B$	✓	7.5	
$x_2, s_1$	$x_1, s_2$	2, 3	$D$	✓	4	
$x_2, s_2$	$x_1, s_1$	5, -6	$E$	✗		
$s_1, s_2$	$x_1, x_2$	1, 2	$C$	✓	8	<b>Optimum</b>

### Exercise 2.2.

1. Consider the following LP:

$$\begin{aligned} \text{max} \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- (a) Express the problem in equation form.
  - (b) Determine all the basic solutions of the problem, and classify them as feasible and infeasible.
  - (c) Use direct substitution in the objective function to determine the optimum basic feasible solution.
  - (d) Verify graphically that the solution obtained in (c) is the optimum LP solution, hence, conclude that the optimum solution can be determined algebraically by considering the basic feasible solutions only.
  - (e) Show how the *infeasible* basic solutions are represented on the graphical solution space.
2. Determine the optimum solution for each of the following LPs by enumerating all the basic solutions.

- (a) **max**  $z = 2x_1 - 4x_2 + 5x_3 - 6x_4$   
**s.t.**  $x_1 + 5x_2 - 2x_3 + 8x_4 \leq 2$   
 $-x_1 + 2x_2 + 3x_3 + 4x_4 \leq 1$   
 $x_1, x_2, x_3, x_4 \geq 0$
- (b) **min**  $w = x_1 + 2x_2 - 3x_3 - 2x_4$   
**s.t.**  $x_1 + 2x_2 - 3x_3 + x_4 = 4$   
 $x_1 + 2x_2 + x_3 + 2x_4 = 4$   
 $x_1, x_2, x_3, x_4 \geq 0$

3. Show algebraically that all the basic solutions of the following LP are infeasible.

$$\begin{array}{ll} \mathbf{max} & z = x_1 + x_2 \\ \mathbf{s.t.} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

4. Consider the following LP:

$$\begin{array}{ll} \mathbf{max} & z = x_1 + 3x_2 \\ \mathbf{s.t.} & x_1 + x_2 \leq 2 \\ & -2x_1 + x_2 \leq 4 \\ & x_1 \text{ urs}, x_2 \geq 0 \end{array}$$

- (a) Determine all the basic feasible solutions of the problem.
- (b) Use direct substitution in the objective function to determine the best basic solution.
- (c) Solve the problem graphically, and verify that the solution obtained in (b) is the optimum.

## 2.4 The Simplex Algorithm

Rather than enumerating *all* the basic solutions (corner points) of the LP problem, as we did in example (2.2), the simplex method investigates only a “select few” of these solutions. This section describes the iterative nature of the method, and provides the computational details of the simplex algorithm.

Before describing the simplex algorithm in general terms, we need to define the concept of an adjacent basic feasible solution.

**Definition 2.3.** For any LP with  $m$  constraints, two basic feasible solutions are said to be *adjacent* if their sets of basic variables have  $m - 1$  basic variables in common.

For example, in Figure 23, two basic feasible solutions will be adjacent if they have  $2 - 1 = 1$  basic variable in common. Thus, the bfs corresponding to point  $B$  in Figure 23 is adjacent to the bfs corresponding to point  $C$  but is not adjacent to bfs  $D$ . Intuitively, two basic feasible solutions are adjacent if they both lie on the same edge of the boundary of the feasible region.

### 2.4.1 Iterative Nature of the Simplex Method

Figure 23 provides the solution space of the LP of example (2.2). For the sake of standardizing the algorithm, the simplex method always starts at the origin where all the decision variables,  $x_i$  are zero. In Figure 23, point  $A$  is the origin ( $x_1 = x_2 = 0$ ) and the associated objective value,  $z$ , is zero. The logical question now is whether an increase in the values of nonbasic  $x_1$  and  $x_2$  above their current zero values can improve (increase) the value of  $z$ . We can answer this question by investigating the objective function:

$$\max z = 2x_1 + 3x_2$$

An increase in  $x_1$  or  $x_2$  (or both) above their current zero values will improve the value of  $z$ . The design of the simplex method does not allow simultaneous increases in variables. Instead, it targets the variables one at a time. The variable slated for increase is the one with the largest rate of improvement in  $z$ . In the present example, the rate of improvement in the value of  $z$  is 2 for  $x_1$  and 3 for  $x_2$ . We thus elect to increase  $x_2$  (the variable with the largest rate of improvement among all nonbasic variables). Figure 23 shows that the value of  $x_2$  must be increased until corner point  $B$  is reached (recall that stopping short of corner point  $B$  is not an option because a candidate for the optimum must be a corner point). At point  $B$ , the simplex method, as will be explained later, will then increase the value of  $x_1$  to reach the improved corner point  $C$ , which is the optimum.

The path of the simplex algorithm always connects adjacent corner points. In the present example the path to the optimum is  $A \rightarrow B \rightarrow C$ . Each corner point along the path is associated with an iteration. It is important to note that the simplex method always moves alongside the edges of the solution space, which means that the method does not cut across the solution space. For example, the simplex algorithm cannot go from  $A$  to  $C$  directly since they are not adjacent.

### 2.4.2 Computational Details of the Simplex Algorithm

We now describe how the simplex algorithm can be used to solve LPs in which the goal is to maximize the objective function. The solution of minimization problems is discussed in Section 2.5. The simplex algorithm proceeds as follows:

**Step 1:** Convert the LP to standard form. Then, write the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in the form

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0.$$

We call this format the *row 0 version* of the objective function (row 0 for short).

**Step 2:** Obtain a bfs (if possible) from the standard form. This is easy if all the constraints are  $\leq$  with nonnegative right-hand sides. Then the slack variable  $s_i$  may be used as the basic variable for row  $i$ . If no bfs is readily apparent, then use the technique discussed in Section 2.6 to find a bfs.

**Step 3:** Determine whether the current bfs is optimal. If all nonbasic variables have nonnegative coefficients in row 0, then the current bfs is optimal. If any variables in row 0 have negative coefficients, then choose the variable with the most negative coefficient in row 0 to enter the basis. We call this variable the *entering* variable.

**Step 4:** If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value. When entering a variable into the basis, compute the ratio

$$\frac{\text{Right-hand side of constraint}}{\text{Coefficient of entering variable in constraint}}$$

for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called the *winner of the ratio test*. The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.

**Step 5:** Use elementary row operations (EROs) to find the new bfs with the better objective function value by making the entering variable a basic variable (has coefficient 1 in pivot row, and 0 in other rows) in the constraint that wins the ratio test. Go back to step 3.

### 2.4.3 Representing the Simplex Tableau

The tabular form of the simplex method records only the essential information:

- the coefficients of the variables,
- the constants on the right-hand sides of the equations,
- the basic variable appearing in each equation.

This saves writing the symbols for the variables in each of the equations, but what is even more important is the fact that it permits highlighting the numbers

involved in arithmetic calculations and recording the computations compactly. For example, the form

$$\begin{array}{rcl} z - 2x_1 - 3x_2 & & = 0 \\ 2x_1 + x_2 + s_1 & & = 4 \\ x_1 + 2x_2 & + s_2 & = 5 \end{array}$$

would be written in abbreviated form as shown in the following table.

Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
$z$	-2	-3	0	0	0
$s_1$	2	1	1	0	4
$s_2$	1	2	0	1	5

The layout of the simplex tableau automatically provides the solution at the starting iteration. The solution starts at the origin  $(x_1, x_2) = (0, 0)$ , thus defining  $(x_1, x_2)$  as the nonbasic variables and  $(s_1, s_2)$  as the basic variables. The associated objective  $z$  and the basic variables  $(s_1, s_2)$  are listed in the leftmost Basic-column. Their values,  $z = 0$ ,  $s_1 = 4$ ,  $s_2 = 5$  appearing in the rightmost Solution-column, are given directly by the right-hand sides of the model's equations (a convenient consequence of starting at the origin). The result can be seen by setting the nonbasic variables  $(x_1, x_2)$  equal to zero in all the equations, and also by noting the special identity-matrix arrangement of the constraint coefficients of the basic variables (all diagonal elements are 1, and all off-diagonal elements are 0).

**Example 2.3.** Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \mathbf{max} & z = 2x_1 + 3x_2 \\ \mathbf{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

**Solution:** By adding slack variables  $s_1$  and  $s_2$ , respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \mathbf{max} & z - 2x_1 - 3x_2 = 0 \\ \mathbf{s.t.} & 2x_1 + x_2 + s_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

		↓						
<b>Iteration [0]</b>	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS		
	$z$	-2	-3	0	0	0		
	$s_1$	2	1	1	0	4	Ratio= $4/1 = 4$	
←	$s_2$	1	2	0	1	5	Ratio= $5/2 = 2.5$	

Iteration [1]	Basic	↓				RHS	
		$x_1$	$x_2$	$s_1$	$s_2$		
		$z$	$-1/2$	$0$	$0$	$3/2$	$15/2$
		$s_1$	$3/2$	$0$	$1$	$-1/2$	$3/2$
←		$x_2$	$1/2$	$1$	$0$	$1/2$	$5/2$

Ratio =  $3/2 \div 3/2 = 1$   
Ratio =  $5/2 \div 1/2 = 5$

Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	Optimal Tableau
		$z$	$0$	$0$	$1/3$	$4/3$	
		$x_1$	$1$	$0$	$2/3$	$-1/3$	
		$x_2$	$0$	$1$	$-1/3$	$2/3$	

$z = 8$   
 $x_1 = 1, x_2 = 2$   
 $s_1 = 0, s_2 = 0$

**Example 2.4.** Solve the following LP problem using the simplex method.

$$\begin{aligned}
 \max \quad & z = 4x_1 + 4x_2 \\
 \text{s.t.} \quad & 6x_1 + 4x_2 \leq 24 \\
 & x_1 + 2x_2 \leq 6 \\
 & -x_1 + x_2 \leq 1 \\
 & x_2 \leq 2 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

**Solution:** By adding slack variables  $s_1$ ,  $s_2$ ,  $s_3$  and  $s_4$ , respectively, we obtain the LP in standard form:

$$\begin{aligned}
 \max \quad & z - 4x_1 - 4x_2 = 0 \\
 \text{s.t.} \quad & 6x_1 + 4x_2 + s_1 = 24 \\
 & x_1 + 2x_2 + s_2 = 6 \\
 & -x_1 + x_2 + s_3 = 1 \\
 & x_2 + s_4 = 2 \\
 & x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
 \end{aligned}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below. Note that we can choose to enter either  $x_1$  or  $x_2$  into the basis. We arbitrarily choose to enter  $x_1$  into basis.

Iteration [0]	Basic	↓						RHS	
		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$		
		$z$	$-4$	$-4$	$0$	$0$	$0$	$0$	
		$s_1$	$6$	$4$	$1$	$0$	$0$	$24$	Ratio = $24/6 = 4$
		$s_2$	$1$	$2$	$0$	$1$	$0$	$6$	Ratio = $6/1 = 6$
		$s_3$	$-1$	$1$	$0$	$0$	$1$	$1$	✗
		$s_4$	$0$	$1$	$0$	$0$	$1$	$2$	✗

		↓							
Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	RHS	
	$z$	0	$-4/3$	$2/3$	0	0	0	16	
	$x_1$	1	$2/3$	$1/6$	0	0	0	4	Ratio= $4 \div 2/3 = 6$
←	$s_2$	0	$4/3$	$-1/6$	1	0	0	2	Ratio= $2 \div 4/3 = 3/2$
	$s_3$	0	$5/3$	$1/6$	0	1	0	5	Ratio= $5 \div 5/3 = 3$
	$s_4$	0	1	0	0	0	1	2	Ratio= $2/1 = 2$

Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	RHS
	$z$	0	0	$1/2$	1	0	0	18
	$x_1$	1	0	$1/4$	$-1/2$	0	0	3
	$x_2$	0	1	$-1/8$	$3/4$	0	0	$3/2$
	$s_3$	0	0	$3/8$	$-5/4$	1	0	$3/2$
	$s_4$	0	0	$1/8$	$-3/4$	0	1	$1/2$

#### Optimal Tableau

$$\begin{aligned}
 z &= 18 \\
 x_1 &= 3, x_2 = 3/2 \\
 s_1 &= 0, s_2 = 0 \\
 s_3 &= 3/2, s_4 = 1/2
 \end{aligned}$$

**Example 2.5.** Solve the following LP problem using the simplex method.

$$\begin{aligned}
 \max \quad & z = x_1 + 3x_2 \\
 \text{s.t.} \quad & x_1 + x_2 \leq 2 \\
 & -x_1 + x_2 \leq 4 \\
 & x_1 \geq 0, x_2 \text{ urs}
 \end{aligned}$$

**Solution:** By assuming  $x_2 = y_1 - y_2$  and then adding slack variables  $s_1$  and  $s_2$ , respectively, we obtain the LP in standard form:

$$\begin{aligned}
 \max \quad & z - x_1 - 3y_1 + 3y_2 = 0 \\
 \text{s.t.} \quad & x_1 + y_1 - y_2 + s_1 = 2 \\
 & -x_1 + y_1 - y_2 + s_2 = 4 \\
 & x_1, y_1, y_2, s_1, s_2 \geq 0
 \end{aligned}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

		↓							
Iteration [0]	Basic	$x_1$	$y_1$	$y_2$	$s_1$	$s_2$		RHS	
	$z$	-1	-3	3	0	0		0	
	$s_1$	1	1	-1	1	0		2	Ratio= $2/1 = 2$
←	$s_2$	-1	1	-1	0	1		4	Ratio= $4/1 = 4$

Iteration [1]	Basic	$x_1$	$y_1$	$y_2$	$s_1$	$s_2$	RHS
	$z$	2	0	0	3	0	6
	$y_1$	1	1	-1	1	0	2
	$s_2$	-2	0	0	-1	1	2

#### Optimal Tableau

$$\begin{aligned}
 z &= 6 \\
 x_1 &= 0, x_2 = 2 \\
 s_1 &= 0, s_2 = 2
 \end{aligned}$$

Note that from the optimal tableau we have  $y_1 = 2$  and  $y_2 = 0$ , so that  $x_2 = y_1 - y_2 = 2 - 0 = 2$ .

### Exercise 2.3.

1. Use the simplex algorithm to solve the following problems.

<p>(a) <b>max</b>      <math>z = 2x_1 + 3x_2</math>  <b>s.t.</b>      <math>x_1 + 2x_2 \leq 6</math>                <math>2x_1 + x_2 \leq 8</math>                <math>x_1, x_2 \geq 0</math></p>	<p>(c) <b>max</b>      <math>z = x_1 - x_2</math>  <b>s.t.</b>      <math>4x_1 + x_2 \leq 100</math>                <math>x_1 + x_2 \leq 80</math>                <math>x_1 \leq 40</math>                <math>x_1, x_2 \geq 0</math></p>
<p>(b) <b>max</b>      <math>z = 2x_1 - x_2 + x_3</math>  <b>s.t.</b>      <math>3x_1 + x_2 + x_3 \leq 60</math>                <math>x_1 - x_2 + 2x_3 \leq 10</math>                <math>x_1 + x_2 - x_3 \leq 20</math>                <math>x_1, x_2, x_3 \geq 0</math></p>	<p>(d) <b>max</b>      <math>z = x_1 + x_2 + x_3</math>  <b>s.t.</b>      <math>x_1 + 2x_2 + 2x_3 \leq 20</math>                <math>2x_1 + x_2 + 2x_3 \leq 20</math>                <math>2x_1 + 2x_2 + x_3 \leq 20</math>                <math>x_1, x_2, x_3 \geq 0</math></p>

2. Solve the following problem by *inspection*, and justify the method of solution in terms of the basic solutions of the simplex method.

$$\begin{array}{ll}
 \mathbf{max} & z = 5x_1 - 6x_2 + 3x_3 - 5x_4 + 12x_5 \\
 \mathbf{s.t.} & x_1 + 3x_2 + 5x_3 + 6x_4 + 3x_5 \leq 30 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{array}$$

## 2.5 Solving Minimization Problem

There are two different ways that the simplex algorithm can be used to solve minimization problems.

**Method (1)** Multiply the objective function for the min problem by  $-1$  and solve the problem as a maximization problem with objective function  $(-w)$ . The optimal solution to the max problem will give you the optimal solution to the min problem where

$$\begin{bmatrix} \text{optimal objective function} \\ \text{value for } \min \text{ problem} \end{bmatrix} = - \begin{bmatrix} \text{optimal objective function} \\ \text{value for } \max \text{ problem} \end{bmatrix}$$

**Example 2.6.** Solve the following LP problem using the simplex method.

$$\begin{array}{ll}
 \mathbf{min} & w = 2x_1 - 3x_2 \\
 \mathbf{s.t.} & x_1 + x_2 \leq 4 \\
 & x_1 - x_2 \leq 6 \\
 & x_1, x_2 \geq 0
 \end{array}$$

**Solution:** The optimal solution to the LP is the point  $(x_1, x_2)$  in the feasible region for the LP that makes  $w = 2x_1 - 3x_2$  the smallest. Equivalently, we may say that the optimal solution to the LP is the point in the feasible region that makes  $z = -w = -2x_1 + 3x_2$  the largest. This means that we can find the optimal solution to the LP by solving:



$$\begin{array}{ll}\text{max} & z = -2x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 6 \\ & x_1, x_2 \geq 0\end{array}$$

By adding slack variables  $s_1$  and  $s_2$ , respectively, we obtain the LP in standard form:

$$\begin{array}{ll}\mathbf{max} & z + 2x_1 - 3x_2 = 0 \\ \mathbf{s.t.} & x_1 + x_2 + s_1 = 4 \\ & x_1 - x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \geq 0\end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

↓

Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
	$z$	2	-3	0	0	0
←	$s_1$	1	1	1	0	4
	$s_2$	1	-1	0	1	6

Ratio =  $4/1 = 4$   
**X**

Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
	$z$	5	0	3	0	12
	$x_2$	1	1	1	0	4
	$s_2$	2	0	1	1	10

**Optimal Tableau**

$w = -z = -12$

$x_1 = 0, x_2 = 4$

$s_1 = 0, s_2 = 10$

**Method (2)** A simple modification of the simplex algorithm can be used to solve min problems directly. Modify *Step 3* of the simplex as follows: If all nonbasic variables in row 0 have nonpositive coefficients, then the current bfs is optimal. If any nonbasic variable in row 0 has a positive coefficient, choose the variable with the “most positive” coefficient in row 0 to enter the basis. This modification of the simplex algorithm works because increasing a nonbasic variable with a positive coefficient in row 0 will decrease  $w$ . If we use this method to solve the LP in example (2.6), then after adding slack variables  $s_1$  and  $s_2$ , respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \min & w - 2x_1 + 3x_2 = 0 \\ \text{s.t.} & x_1 + x_2 + s_1 = 4 \\ & x_1 - x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below. Note that, because  $x_2$  has the most positive coefficient in row 0, we enter  $x_2$  into the basis.

Iteration [0]	Basic	↓				RHS	
		$x_1$	$x_2$	$s_1$	$s_2$		
		$w$	-2	3	0	0	
		$s_1$	1	1	1	0	
←	$s_2$	1	-1	0	1	6	Ratio = $4/1 = 4$
							<b>X</b>
Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	<b>Optimal Tableau</b> $w = -12$ $x_1 = 0, x_2 = 4$ $s_1 = 0, s_2 = 10$
		$w$	-5	0	-3		
		$x_2$	1	1	1		
		$s_2$	2	0	1		

**Exercise 2.4.** Use the simplex algorithm to solve the following problems.

1. **min**  $w = 4x_1 - x_2$   
**s.t.**  $2x_1 + x_2 \leq 8$   
 $x_2 \leq 5$   
 $x_1 - x_2 \leq 4$   
 $x_1, x_2 \geq 0$
2. **min**  $w = x_1 - x_2$   
**s.t.**  $x_1 - x_2 \leq 1$   
 $x_1 + x_2 \leq 2$   
 $x_1, x_2 \geq 0$
3. **min**  $w = 2x_1 - 5x_2$   
**s.t.**  $3x_1 + 8x_2 \leq 12$   
 $2x_1 + 3x_2 \leq 6$   
 $x_1, x_2 \geq 0$
4. **min**  $w = -3x_1 + 8x_2$   
**s.t.**  $4x_1 + 2x_2 \leq 12$   
 $2x_1 + 3x_2 \leq 6$   
 $x_1, x_2 \geq 0$
5. **min**  $w = 5x_1 + 4x_2 + 6x_3 - 8x_4$   
**s.t.**  $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40$   
 $2x_1 - 2x_2 + x_3 + 2x_4 \leq 8$   
 $4x_1 - 2x_2 + x_3 - x_4 \leq 10$   
 $x_1, x_2, x_3, x_4 \geq 0$

## 2.6 Artificial Starting Solution and the Big $M$ -Method

Recall that the simplex algorithm requires a starting bfs. In all the problems we have solved so far, we found a starting bfs by using the slack variables as our basic variables. If an LP has any ( $\geq$ ) or ( $=$ ) constraints, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the *Big  $M$ -method* may be used to solve the problem. The Big  $M$ -method is a version of the simplex algorithm that first finds a bfs by adding “artificial” variables to the problem. The objective function of the original LP must be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

The big  $M$ -method starts with the LP in equation form. If equation  $i$  does not have a slack (or a variable that can play the role of a slack), an artificial variable,  $a_i$ , is added to form a starting solution similar to the all-slack basic solution. However, because the artificial variables are not part of the original problem, a modeling “trick” is needed to force them to zero value by the time

the optimum iteration is reached (assuming the problem has a feasible solution). The desired goal is achieved by assigning a penalty defined as:

$$\begin{array}{l} \text{Artificial variable objective} \\ \text{function coefficient} \end{array} = \begin{cases} -M & \text{in max problems} \\ M & \text{in min problems} \end{cases}$$

where  $M$  is a sufficiently large positive value (mathematically,  $M \rightarrow \infty$ ).

**Example 2.7.** Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \min & w = 4x_1 + x_2 \\ \text{s.t.} & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

**Solution:** To convert the constraint to equations, use  $e_2$  as a surplus in the second constraint and  $s_3$  as a slack in the third constraint.

$$\begin{array}{lll} 3x_1 + x_2 & & = 3 \\ 4x_1 + 3x_2 - e_2 & & = 6 \\ x_1 + 2x_2 & + s_3 & = 4 \end{array}$$

The third equation has its slack variable,  $s_3$ , but the first and second equations do not. Thus, we add the artificial variables  $a_1$  and  $a_2$  in the first two equations and penalize them in the objective function with  $Ma_1 + Ma_2$  (because we are minimizing). The resulting LP becomes

$$\begin{array}{ll} \min & w = 4x_1 + x_2 + Ma_1 + Ma_2 \\ \text{s.t.} & 3x_1 + x_2 + a_1 = 3 \\ & 4x_1 + 3x_2 - e_2 + a_2 = 6 \\ & x_1 + 2x_2 + s_3 = 4 \\ & x_1, x_2, s_3, e_2, a_1, a_2 \geq 0 \end{array}$$

After writing the objective function as  $w - 4x_1 - x_2 - Ma_1 - Ma_2 = 0$ , the initial tableau will be

Iteration [0]	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS
	$w$	-4	-1	0	0	$-M$	$-M$	0
	$a_1$	3	1	0	0	1	0	3
	$a_2$	4	3	0	-1	0	1	6
	$s_3$	1	2	1	0	0	0	4

Before proceeding with the simplex method computations, row 0 must be made consistent with the rest of the tableau. The right-hand side of row 0 in the tableau currently shows  $w = 0$ . However, given the nonbasic solution  $x_1 = x_2 = e_2 = 0$ , the current basic solution is  $a_1 = 3$ ,  $a_2 = 6$ , and  $s_3 = 4$  yields

$$w = (4 \times 0) + (1 \times 0) + (3 \times M) + (6 \times M) = 9M \neq 0.$$

The inconsistency stems from the fact that  $a_1$  and  $a_2$  have nonzero coefficients in row 0. To eliminate the inconsistency, we use EROs. The modified tableau thus becomes (verify!):

Iteration [0]	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS
	$w$	$7M - 4$	$4M - 1$	0	$-M$	0	0	$9M$
	$a_1$	3	1	0	0	1	0	3
	$a_2$	4	3	0	-1	0	1	6
	$s_3$	1	2	1	0	0	0	4

The last tableau is ready for the application of the simplex optimality and the feasibility conditions. Because the objective function is minimized, the variable  $x_1$  having the most positive coefficient in the row 0 enters the solution. The minimum ratio of the feasibility condition specifies  $a_1$  as the leaving variable. All tableaus until the optimal solution is reached are shown below.

		$\downarrow$							
<b>Iteration [0]</b>	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS	
	$w$	$7M - 4$	$4M - 1$	0	$-M$	0	0	$9M$	
$\leftarrow$	$a_1$	3	1	0	0	1	0	3	Ratio= $3/3 = 1$
	$a_2$	4	3	0	$-1$	0	1	6	Ratio= $6/4 = 3/2$
	$s_3$	1	2	1	0	0	0	4	Ratio= $4/2 = 2$

		↓							
Iteration [1]	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS	
	$w$	0	$(1+5M)/3$	0	$-M$	$(4-7M)/3$	0	$4 + 2M$	
←	$x_1$	1	$1/3$	0	0	$1/3$	0	1	Ratio= $1 \div 1/3 = 3$
	$a_2$	0	$5/3$	0	-1	$-4/3$	1	2	Ratio= $2 \div 5/3 = 6/5$
	$s_3$	0	$5/3$	1	0	$-1/3$	0	3	Ratio= $3 \div 5/3 = 9/5$

		↓							
Iteration [2]	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS	
	$w$	0	0	0	$1/5$	$8/5 - M$	$-1/5 - M$	$18/5$	
	$x_1$	1	0	0	$1/5$	$3/5$	$-1/5$	$3/5$	Ratio = $3/5 \div 1/5 = 3$
	$x_2$	0	1	0	$-3/5$	$-4/5$	$3/5$	$6/5$	<b>X</b>
←	$s_3$	0	0	1	1	1	-1	1	Ratio = $1/1 = 1$

Iteration [3]	Basic	$x_1$	$x_2$	$s_3$	$e_2$	$a_1$	$a_2$	RHS	Optimal Tableau
$w$		0	0	$-1/5$	0	$7/5 - M$	$-M$	$17/5$	
$x_1$		1	0	$-1/5$	0	$2/5$	0	$2/5$	
$x_2$		0	1	$3/5$	0	$-1/5$	0	$9/5$	
$e_2$		0	0	1	1	1	$-1$	1	

**Note 9.** From a computational standpoint, solving the problem on the computer requires replacing  $M$  with a *sufficiently large* numeric value. The result is an unnecessary layer of computational difficulty that can be avoided by substituting an appropriate numeric value for  $M$  (which is what we would do anyway if we

use the computer). We break away from the long tradition of manipulating  $M$  algebraically and use a numerical substitution instead. The intent is to simplify the presentation without losing substance. What value of  $M$  should we use? The answer depends on the data of the original LP. Recall that the penalty  $M$  must be sufficiently large relative to the original objective coefficients to force the artificial variables to be zero (which happens only if a feasible solution exists). At the same time, since computers are the main tool for solving LPs,  $M$  should not be unnecessarily too large, as this may lead to serious round-off error. In the present example, the objective coefficients of  $x_1$  and  $x_2$  are 2 and 1, respectively, and it appears reasonable to set  $M = 100$ .

**Example 2.8.** Solve the following LP problem using the simplex method.

$$\begin{array}{ll}\mathbf{max} & z = 2x_1 + x_2 \\ \mathbf{s.t.} & x_1 + x_2 \leq 10 \\ & -x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** To convert the constraint to equations, use  $s_1$  as a slack in the first constraint and  $e_2$  as a surplus in the second constraint.

$$\begin{array}{rclcl}x_1 + x_2 + s_1 & & & = & 10 \\ -x_1 + x_2 & & - e_2 & = & 2\end{array}$$

We add the artificial variables  $a_2$  in the second equation and penalize it in the objective function with  $-Ma_2 = -100a_2$  (because we are maximizing). The resulting LP becomes

$$\begin{array}{ll}\mathbf{max} & z = 2x_1 + x_2 - 100a_2 \\ \mathbf{s.t.} & x_1 + x_2 + s_1 = 10 \\ & -x_1 + x_2 - e_2 + a_2 = 2 \\ & x_1, x_2, s_1, e_2, a_2 \geq 0\end{array}$$

After writing the objective function as  $z - 2x_1 - x_2 + 100a_2 = 0$ , the initial tableau will be

Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$e_2$	$a_2$	RHS
	$z$	-2	-1	0	0	100	0
	$s_1$	1	1	1	0	0	10
	$a_2$	-1	1	0	-1	1	2

Before proceeding with the simplex method computations, row 0 must be made consistent with the rest of the tableau. The inconsistency stems from the fact that  $a_2$  has nonzero coefficients in row 0. To eliminate the inconsistency, we use EROs. The modified tableau and all other tableaus until the optimal solution is reached are:

Iteration [0]	Basic	↓					RHS	
		$x_1$	$x_2$	$s_1$	$e_2$	$a_2$		
		$z$	98	-101	0	100	0	-200
		$s_1$	1	1	1	0	0	10
←	$a_2$	-1	1	0	-1	1	2	Ratio = $10/1 = 10$ Ratio = $2/1 = 2$

Iteration [1]	Basic	↓					RHS	
		$x_1$	$x_2$	$s_1$	$e_2$	$a_2$		
		$z$	-3	0	0	-1	101	2
		$s_1$	2	0	1	1	-1	8
←	$x_2$	-1	1	0	-1	1	2	Ratio = $8/2 = 4$ ✗

Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$e_2$	$a_2$	RHS	Optimal Tableau $z = 14$ $x_1 = 4, x_2 = 6$ $s_1 = e_2 = a_2 = 0$
		$z$	0	0	$3/2$	$1/2$	$199/2$	
		$x_1$	1	0	$1/2$	$1/2$	$-1/2$	
		$x_2$	0	1	$1/2$	$-1/2$	$1/2$	

**Example 2.9.** Consider the problem.

$$\begin{aligned}
 \max \quad & z = 2x_1 + 4x_2 + 4x_3 - 3x_4 \\
 \text{s.t.} \quad & x_1 + x_2 + x_3 = 4 \\
 & x_1 + 4x_2 + x_4 = 8 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

The variables  $x_3$  and  $x_4$  play the role of slack variables. So, without using any artificial variables, solve the problem with  $x_3$  and  $x_4$  as the starting basic variables.

**Solution:** The main difference here from the usual simplex is that  $x_3$  and  $x_4$  have nonzero objective coefficients in row 0:  $z - 2x_1 - 4x_2 - 4x_3 + 3x_4 = 0$ . To eliminate their coefficients, we use EROs. The initial tableaux and all following tableaux until the optimal solution is reached are shown below.

Iteration [0]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	RHS
	$z$	-2	-4	-4	3	0
	$x_3$	1	1	1	0	4
	$x_4$	1	4	0	1	8
↓						
Iteration [0]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	RHS
	$z$	-1	-12	0	0	-8
	$x_3$	1	1	1	0	4
	$x_4$	1	4	0	1	8
←						Ratio= $4/1 = 4$ Ratio= $8/4 = 2$

Iteration [1]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	RHS	Optimal Tableau $z = 16$ $x_1 = 0, x_2 = 2$ $x_3 = 2, x_4 = 0$
		$z$	2	0	0	3	16
		$x_3$	$3/4$	0	1	$-1/4$	2
		$x_2$	$1/4$	1	0	$1/4$	2

### Exercise 2.5.

1. Use the Big  $M$ -method to solve the following LPs:

<p>(a) <b>min</b>     <math>w = 4x_1 + 4x_2 + x_3</math> <b>s.t.</b>     <math>x_1 + x_2 + x_3 \leq 2</math>             <math>2x_1 + x_2 \leq 3</math>             <math>2x_1 + x_2 + 3x_3 \geq 3</math>             <math>x_1, x_2, x_3 \geq 0</math></p>	<p>(c) <b>min</b>     <math>w = 2x_1 + 3x_2</math> <b>s.t.</b>     <math>2x_1 + x_2 \geq 4</math>             <math>x_1 - x_2 \geq -1</math>             <math>x_1, x_2 \geq 0</math></p>
<p>(b) <b>min</b>     <math>w = x_1 + x_2</math> <b>s.t.</b>     <math>2x_1 + x_2 + x_3 = 4</math>             <math>x_1 + x_2 + 2x_3 \leq 2</math>             <math>x_1, x_2, x_3 \geq 0</math></p>	<p>(d) <b>max</b>     <math>z = 3x_1 + x_2</math> <b>s.t.</b>     <math>2x_1 + x_2 \leq 4</math>             <math>x_1 + x_2 = 3</math>             <math>x_1, x_2 \geq 0</math></p>

2. Solve the following problem using  $x_3$  and  $x_4$  as starting basic feasible variables. As in example (2.9), do not use any artificial variables.

$$\begin{array}{ll} \text{min} & z = 3x_1 + 2x_2 + 3x_3 + 2x_4 \\ \text{s.t.} & x_1 + 4x_2 + x_3 \geq 14 \\ & 2x_1 + x_2 + x_4 \geq 20 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

3. Consider the problem

$$\begin{array}{ll} \text{max} & z = x_1 + 5x_2 + 3x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 = 6 \\ & 2x_1 - x_2 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The variable  $x_3$  plays the role of a slack. Thus, no artificial variable is needed in the first constraint. In the second constraint, an artificial variable,  $a_2$ , is needed. Solve the problem using  $x_3$  and  $a_2$  as the starting variables.

## 2.7 Special Cases in the Simplex Method

This section considers four special cases that arise in the use of the simplex method.

### 2.7.1 Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one basic variable will be zero in the next iteration, and the new solution is said to be degenerate. This situation may reveal that the model has at least one redundant constraint.

**Definition 2.4.** An LP is *degenerate* if it has at least one bfs in which a basic variable is equal to zero.

If one of these degenerate basic variables retains its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero, so the value of the objective function must remain unchanged. However, if the objective function may remain the same rather than change at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually changing the objective function toward an optimal solution. This occurrence is called *cycling*.

**Example 2.10.** Solve the following LP problem.

$$\begin{array}{ll}\max & z = 3x_1 + 9x_2 \\ \text{s.t.} & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** By adding slack variables  $s_1$  and  $s_2$ , we obtain the LP in standard form

$$\begin{array}{ll}\max & z - 3x_1 - 9x_2 = 0 \\ \text{s.t.} & x_1 + 4x_2 + s_1 = 8 \\ & x_1 + 2x_2 + s_2 = 4 \\ & x_1, x_2, s_1, s_2 \geq 0\end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

		↓					
Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
	$z$	-3	-9	0	0	0	
←	$s_1$	1	4	1	0	8	Ratio = $8/4 = 2$
	$s_2$	1	2	0	1	4	Ratio = $4/2 = 2$

In iteration 0,  $s_1$  and  $s_2$  tie for the leaving variable, leading to degeneracy in iteration 1 because the basic variable  $s_2$  assumes a zero value.

		↓					
Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
	$z$	$-3/4$	0	$9/4$	0	18	
	$x_2$	$1/4$	1	$1/4$	0	2	Ratio = $2 \div 1/4 = 8$
←	$s_2$	$1/2$	0	$-1/2$	1	0	Ratio = $0 \div 1/2 = 0$

Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
	$z$	0	0	$3/2$	$3/2$	18
	$x_2$	0	1	$1/2$	$-1/2$	2
	$x_1$	1	0	-1	2	0

**Optimal Tableau**

$z = 18$   
 $x_1 = 0, x_2 = 2$   
 $s_1 = 0, s_2 = 0$



The following example illustrates the occurrence of cycling in the simplex iterations and the possibility that the algorithm may never converge to the optimum solution.

**Example 2.11.** This example was authored by E.M. Beale<sup>1</sup>. Consider the following LP:

$$\begin{aligned}
 \max \quad & C = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4 \\
 \text{s.t.} \quad & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \leq 0 \\
 & \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \leq 0 \\
 & x_3 \leq 1 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

Actually, the optimal solution of this example is  $C = \frac{1}{20}$  when  $x_1 = \frac{1}{25}$ ,  $x_3 = 1$ , and  $x_2 = x_4 = 0$ . However, in order to solve this LP using the Simplex algorithm, we write it in standard form as follows.

$$\begin{aligned}
 \max \quad & C - \frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 = 0 \\
 \text{s.t.} \quad & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + s_1 = 0 \\
 & \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + s_2 = 0 \\
 & x_3 + s_3 = 1 \\
 & x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0
 \end{aligned}$$

Let us start applying the Simplex algorithm, and see what will happen through the iterations.

		↓								
Iteration [0]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	$-3/4$	150	$-1/50$	6	0	0	0	0	
	← $s_1$	$1/4$	-60	$-1/25$	9	1	0	0	0	Ratio= 0
	$s_2$	$1/2$	-90	$-1/50$	3	0	1	0	0	Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1	✗
		↓								
Iteration [1]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	0	-30	$-7/50$	33	3	0	0	0	
	$x_1$	1	-240	$-4/25$	36	4	0	0	0	✗
	← $s_2$	0	30	$3/50$	-15	-2	1	0	0	Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1	✗

<sup>1</sup>Saul I. Gass, Sasirekha Vinjamuri. **Cycling in linear programming problems**. Computers & Operations Research 31 (2004)

↓									
<b>Iteration [2]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS
	$C$	0	0	$-2/25$	18	1	1	0	0
←	$x_1$	1	0	$8/25$	$-84$	$-12$	8	0	0 Ratio= 0
	$x_2$	0	1	$1/500$	$-1/2$	$-1/15$	$1/30$	0	0 Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1 Ratio= 1
↓									
<b>Iteration [3]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS
	$C$	$1/4$	0	0	$-3$	$-2$	3	0	0
	$x_3$	$25/8$	0	1	$-525/2$	$-75/2$	25	0	0 ✗
←	$x_2$	$-1/160$	1	0	$1/40$	$1/120$	$-1/60$	0	0 Ratio= 0
	$s_3$	$-25/8$	0	0	$525/2$	$75/2$	$-25$	1	1 Ratio= $2/525$
↓									
<b>Iteration [4]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS
	$C$	$-1/2$	120	0	0	$-1$	1	0	0
←	$x_3$	$-125/2$	10500	1	0	50	$-150$	0	0 Ratio= 0
	$x_4$	$-1/4$	40	0	1	$1/3$	$-2/3$	0	0 Ratio= 0
	$s_3$	$125/2$	$-10500$	0	0	$-50$	150	1	1 ✗
↓									
<b>Iteration [5]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS
	$C$	$-7/4$	330	$1/50$	0	0	$-2$	0	0
	$s_1$	$-5/4$	210	$1/50$	0	1	$-3$	0	0 ✗
←	$x_4$	$1/6$	$-30$	$-1/150$	1	0	$1/3$	0	0 Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1 ✗
↓									
<b>Iteration [6]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS
	$C$	$-3/4$	150	$-1/50$	6	0	0	0	0
	$s_1$	$1/4$	$-60$	$-1/25$	9	1	0	0	0
	$s_2$	$1/2$	$-90$	$-1/50$	3	0	1	0	0
	$s_3$	0	0	1	0	0	0	1	1

**We back to iteration [0] and we stuck in a cycle**

**Note 10.** There are several ways to solve the LP problem in example (2.11). We review these methods as follows.

1. **Computer Systems:** like Excel Solver, LINDO and Mathematica.
2. **Convert all the coefficients in the constraints to integer values by using proper multiples:**<sup>2</sup> this can be done by multiplying the first constraint in the original LP by  $\text{lcm}(4, 25) = 100$  and the second constraint by  $\text{lcm}(2, 50) = 50$ . Then we write the LP in standard form.

<sup>2</sup>Hamdy A. Taha, **Operations Research: An Introduction, 9th Edition**, Prentice Hall. 2011. Call number in PU library: 658.4034 TAH.

$$\begin{array}{ll}
\max & C - \frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 = 0 \\
\text{s.t.} & 25x_1 - 6000x_2 - 4x_3 + 900x_4 + s_1 = 0 \\
& 25x_1 - 4500x_2 - x_3 + 150x_4 + s_2 = 0 \\
& x_3 + s_3 = 1 \\
& x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0
\end{array}$$

Now we apply the Simplex algorithm.

		↓								
Iteration [0]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	$-3/4$	150	$-1/50$	6	0	0	0	0	
	← $s_1$	25	-6000	-4	900	1	0	0	0	Ratio= 0
	$s_2$	25	-4500	-1	150	0	1	0	0	Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1	✗
↓										
Iteration [1]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	0	-30	$-7/50$	33	$3/100$	0	0	0	
	$x_1$	1	-240	$-4/25$	36	$1/25$	0	0	0	✗
	← $s_2$	0	1500	3	-750	-1	1	0	0	Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1	✗
↓										
Iteration [2]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	0	0	$-2/25$	18	$1/100$	$1/50$	0	0	
	$x_1$	1	0	$8/25$	-84	$-3/25$	$4/25$	0	0	Ratio= 0
	← $x_2$	0	1	$1/500$	$-1/2$	$-1/1500$	$1/1500$	0	0	Ratio= 0
	$s_3$	0	0	1	0	0	0	1	1	Ratio= 1
↓										
Iteration [3]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	0	40	0	-2	$-1/60$	$7/150$	0	0	
	$x_1$	1	-160	0	-4	$-1/75$	$4/75$	0	0	✗
	$x_3$	0	500	1	-250	$-1/3$	$1/3$	0	0	✗
	← $s_3$	0	-500	0	250	$1/3$	$-1/3$	1	1	Ratio= $1/250$
↓										
Iteration [4]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
	$C$	0	36	0	0	$-7/500$	$11/250$	$1/125$	$1/125$	
	$x_1$	1	-168	0	0	$-1/125$	$6/125$	$2/125$	$2/125$	✗
	$x_3$	0	0	1	0	0	0	1	1	✗
	← $x_4$	0	-2	0	1	$1/750$	$-1/750$	$1/250$	$1/250$	Ratio= 3

<b>Iteration [5]</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	<b>Optimal Tableau</b>
	$C$	0	15	0	$21/2$	0	$3/100$	$1/20$	$1/20$	
	$x_1$	1	-180	0	6	0	$1/25$	$1/25$	$1/25$	
	$x_3$	0	0	1	0	0	0	1	1	
	$s_1$	0	-1500	0	750	1	-1	3	3	

### 3. Bland's Rule for selecting entering and leaving variables<sup>3</sup>.

- For the entering basic variable: Of all negative coefficients in the objective row (Row 0), choose the one with smallest subscript.
- For the departing basic variable: When there is a tie between one or more ratios computed, choose the candidate for departing basic variable that has the smallest subscript.

When we use Bland's rule to solve the LP in example (2.11), we name the slack variables  $s_1, s_2, s_3$  as  $x_5, x_6, x_7$  respectively. The iterations will be as follows.

<b>Iteration [0]</b>	Basic	↓							RHS	
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$		
		$C$	$-3/4$	150	$-1/50$	6	0	0	0	0
		← $x_5$	$1/4$	-60	$-1/25$	9	1	0	0	0    Ratio= 0
		$x_6$	$1/2$	-90	$-1/50$	3	0	1	0	0    Ratio= 0
	$x_7$	0	0	1	0	0	0	0	1	1    ✗

<b>Iteration [1]</b>	Basic	↓							RHS	
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$		
		$C$	0	-30	$-7/50$	33	3	0	0	0
		← $x_1$	1	-240	$-4/25$	36	4	0	0	0    ✗
		$x_6$	0	30	$3/50$	-15	-2	1	0	0    Ratio= 0
	$x_7$	0	0	1	0	0	0	0	1	1    ✗

<b>Iteration [2]</b>	Basic	↓							RHS	
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$		
		$C$	0	0	$-2/25$	18	1	1	0	0
		← $x_1$	1	0	$8/25$	-84	-12	8	0	0    Ratio= 0
		$x_2$	0	1	$1/500$	$-1/2$	$-1/15$	$1/30$	0	0    Ratio= 0
	$x_7$	0	0	1	0	0	0	0	1	1    Ratio= 1

<sup>3</sup>James Calvert and William Voxman, **Linear Programming, 1st Edition**, Harcourt Brace Jovanovich Publishers, 1989.

Iteration [3]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	RHS		
		$C$	$1/4$	0	0	-3	-2	3	0	0	
		$x_3$	$25/8$	0	1	$-525/2$	$-75/2$	25	0	0	✗
		$x_2$	$-1/160$	1	0	$1/40$	$1/120$	$-1/60$	0	0	Ratio= 0
		$x_7$	$-25/8$	0	0	$525/2$	$75/2$	-25	1	1	Ratio= $2/525$
↓											
Iteration [4]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	RHS		
		$C$	$-1/2$	120	0	0	-1	1	0	0	
		$x_3$	$-125/2$	10500	1	0	50	-150	0	0	✗
		$x_4$	$-1/4$	40	0	1	$1/3$	$-2/3$	0	0	✗
		$x_7$	$125/2$	-10500	0	0	-50	150	1	1	Ratio= $2/125$
↓											
Iteration [5]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	RHS		
		$C$	0	36	0	0	$-7/5$	$11/5$	$1/125$	$1/125$	
		$x_3$	0	0	1	0	0	0	1	1	✗
		$x_4$	0	-2	0	1	$2/15$	$-1/15$	$1/250$	$1/250$	Ratio= $100/3$
		$x_1$	1	-168	0	0	$-4/5$	$12/5$	$2/125$	$2/125$	✗
↓											
Iteration [6]	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	RHS	Optimal Tableau	
		$C$	0	15	0	$21/2$	0	$3/2$	$1/20$		$1/20$
		$x_3$	0	0	1	0	0	0	1		1
		$x_5$	0	-15	0	$15/2$	1	$-1/2$	$3/100$		$3/100$
		$x_1$	1	-180	0	6	0	2	$1/25$		$1/25$

**Optimal  
Tableau**

#### 4. Lexicographic Rule for selecting an exiting variable<sup>4</sup>.

Given a basic feasible solution with basis  $B$ , suppose that the nonbasic variable  $x_k$  is chosen to enter the basis (the most negative value in Row 0 for maximization LP). The index  $r$  of the variable  $x_B$  leaving the basis is determined as follows. Let

$$I_0 = \left\{ r : \frac{b_r}{y_{rk}} = \min_{0 \leq i \leq m} \left\{ \frac{b_i}{y_{ik}} : y_{ik} > 0 \right\} \right\}$$

If  $I_0$  is a singleton, namely  $I_0 = \{r\}$ , then  $x_{B_r}$  leaves the basis. Otherwise, form  $I_1$  as follows:

$$I_1 = \left\{ r : \frac{y_{r1}}{y_{rk}} = \min_{i \in I_0} \left\{ \frac{y_{i1}}{y_{ik}} \right\} \right\}$$

where  $y_{*1}$  is the first column of the  $m \times m$  identity matrix. If  $I_1$  is singleton, namely,  $I_1 = \{r\}$ , then  $x_{B_r}$  leaves the basis. Otherwise, form

<sup>4</sup> Mokhtar S. Bazaraa, John J. Jarvis, Hanif D. Sher, **Linear Programming and Network Flows, 4th Edition**, John Wiley & Sons, Inc. 2010. Call number in PU library: 519.72 BAZ

$I_2$ , where, in general,  $I_j$  is formed from  $I_{j-1}$  as follows:

$$I_j = \left\{ r : \frac{y_{rj}}{y_{rk}} = \min_{i \in I_{j-1}} \left\{ \frac{y_{ij}}{y_{ik}} \right\} \right\}$$

where  $y_{*j}$  is the  $j$ th column of the  $m \times m$  identity matrix. Eventually, for some  $j \leq m$ ,  $I_j$  will be a singleton. If  $I_j = \{r\}$ , then  $x_{B_r}$  leaves the basis.

		↓									
Iteration [0]		Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
		$C$	$-3/4$	150	$-1/50$	6	0	0	0	0	
		$s_1$	$1/4$	-60	$-1/25$	9	1	0	0	0	Ratio= 0
←		$s_2$	$1/2$	-90	$-1/50$	3	0	1	0	0	Ratio= 0
		$s_3$	0	0	1	0	0	0	1	1	✗

Here,  $I_0 = \{1, 2\}$ , and then  $I_1 = \{2\}$ , and therefore  $x_{B_2} = s_2$  leaves the basis.

		↓									
Iteration [1]		Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	
		$C$	0	15	$-1/20$	$21/2$	0	$3/2$	0	0	
		$s_1$	0	-15	$-3/100$	$15/2$	1	$-1/2$	0	0	✗
		$x_1$	1	-180	$-1/25$	6	0	2	0	0	✗
←		$s_3$	0	0	1	0	0	0	1	1	Ratio= 1

Here,  $I_0 = \{3\}$ . Therefore,  $x_{B_3} = s_3$  leaves.

Iteration [2]		Basic	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	RHS	Optimal Tableau
		$C$	0	15	0	$21/2$	0	$3/2$	$1/20$	$1/20$	
		$s_1$	0	-15	0	$15/2$	1	$-1/2$	$3/100$	$3/100$	
		$x_1$	1	-180	0	6	0	2	$1/25$	$1/25$	
		$x_3$	0	0	1	0	0	0	1	1	

### 2.7.2 Alternative Optima

Recall from example (1.13) of Section 1.4 that for some LPs, more than one extreme point is optimal. If an LP has more than one optimal solution, then we say that it has multiple or alternative optimal solutions. An LP problem may have an infinite number of alternative optima when the objective function is parallel to a nonredundant binding constraint. The existence of alternative can be detected in the optimal tableau by examining row 0 coefficients of the

nonbasic variables. The zero coefficient of nonbasic  $x_j$  indicates that  $x_j$  can be made basic, altering the values of the basic variables without changing the value of  $z$ .

In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. If the example represents a product-mix situation, it may be advantageous to market two products instead of one.

**Example 2.12.** Solve the following LP problem.

$$\begin{array}{ll}\max & z = 2x_1 + 4x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** By adding slack variables  $s_1$  and  $s_2$ , we obtain the LP in standard form

$$\begin{array}{ll}\max & z - 2x_1 - 4x_2 = 0 \\ \text{s.t.} & x_1 + 2x_2 + s_1 = 5 \\ & x_1 + x_2 + s_2 = 4 \\ & x_1, x_2, s_1, s_2 \geq 0\end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

		↓						
<b>Iteration [0]</b>		Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
		$z$	-2	-4	0	0	0	
←		$s_1$	1	2	1	0	5	Ratio = $5/2$
		$s_2$	1	1	0	1	4	Ratio = $4/1 = 4$
		↓						
<b>Iteration [1]</b>		Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
		$z$	0	0	2	0	10	
		$x_2$	$1/2$	1	$1/2$	0	$5/2$	Ratio = $5/2 \div 1/2 = 5$
←		$s_2$	$1/2$	0	$-1/2$	1	$3/2$	Ratio = $3/2 \div 3/2 = 3$
		↓						
<b>Iteration [2]</b>		Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
		$z$	0	0	2	0	10	
		$x_2$	0	1	1	-1	1	
		$x_1$	1	0	-1	2	3	

**Optimal  
Tableau**

**Alternative  
Optima**

Mathematically, we can determine all the points  $(x_1, x_2)$  on the line segment joining the optimal solutions  $(0, \frac{5}{2})$  and  $(3, 1)$  as follows:

$$\left. \begin{array}{l} x_1 = t(0) + (1-t)(3) = 3 - 3t \\ x_2 = t\left(\frac{5}{2}\right) + (1-t)(1) = 1 + \frac{3t}{2} \end{array} \right\} ; 0 \leq t \leq 1$$

### 2.7.3 Unbounded Solutions

In some LP models, as in example (1.15) of Section 1.4, the solution space is unbounded in at least one variable, meaning that variables may be increased indefinitely without violating any of the constraints. The associated objective value may also be unbounded in this case. An unbounded LP for a max problem occurs when a variable with a negative coefficient (positive for min LP) in row 0 has a nonpositive coefficient in each constraint.

An unbounded solution space may signal that the model is poorly constructed. The most likely irregularity in such models is that some key constraints have not been accounted for. Another possibility is that estimates of the constraint coefficients may not be accurate.

**Example 2.13.** Solve the following LP problem.

$$\begin{array}{ll}\text{max} & z = 2x_1 + x_2 \\ \text{s.t.} & x_1 - x_2 \leq 10 \\ & 2x_1 \leq 40 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** By adding slack variables  $s_1$  and  $s_2$ , we obtain the LP in standard form

$$\begin{array}{ll}\text{max} & z - 2x_1 - x_2 = 0 \\ \text{s.t.} & x_1 - x_2 + s_1 = 10 \\ & 2x_1 + s_2 = 40 \\ & x_1, x_2, s_1, s_2 \geq 0\end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
	$z$	-2	-1	0	0	0
	$s_1$	1	-1	1	0	10
	$s_2$	2	0	0	1	40

In the starting tableau, both  $x_1$  and  $x_2$  have negative  $z$ -equation coefficients, meaning that an increase in their values will increase the objective value. Although  $x_1$  should be the entering variable (it has the most negative  $z$ -coefficient), we note that all the constraint coefficients under  $x_2$  are  $\leq 0$ , meaning that  $x_2$  can be increased indefinitely without violating any of the constraints. The result is that  $z$  can be increased indefinitely.

### 2.7.4 Nonexisting (or Infeasible) Solutions

LP models with inconsistent constraints have no feasible solution, see example (1.14) of Section 1.4. This situation does not occur if all the constraints are





Optimum iteration 1 shows that the artificial variable  $a_2$  is positive ( $= 4$ ), meaning that the LP is infeasible. The result is what we may call a *pseudo-optimal* solution.

### Exercise 2.6.

1. Consider the following LP:

$$\begin{array}{ll}\mathbf{max} & z = 3x_1 + 2x_2 \\ \mathbf{s.t.} & 4x_1 - x_2 \leq 4 \\ & 4x_1 + 3x_2 \leq 6 \\ & 4x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0\end{array}$$

- (a) Show that the associated simplex iterations are temporarily degenerate. How many iterations are needed to reach the optimum?
  - (b) Verify the result by solving the problem graphically.
  - (c) Interchange constraints (1) and (3) and resolve the problem. How many iterations are needed to solve the problem?
2. Solve the following problem, using the lexicographic rule for noncycling. Repeat using Bland's Rule:

$$\begin{array}{ll}\mathbf{max} & z = x_1 + 2x_2 + x_3 \\ \mathbf{s.t.} & x_1 + 4x_2 + 3x_3 \leq 4 \\ & -x_1 + x_2 + 4x_3 \leq 1 \\ & x_1 + 3x_2 + x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

3. For the following LP, identify three alternative optimal basic solutions.

$$\begin{array}{ll}\mathbf{max} & z = x_1 + 2x_2 + 3x_3 \\ \mathbf{s.t.} & x_1 + 2x_2 + 3x_3 \leq 10 \\ & x_1 + x_2 \leq 5 \\ & x_1 \leq 1 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

4. Solve the following LP:

$$\begin{array}{ll}\mathbf{max} & z = 2x_1 - x_2 + 3x_3 \\ \mathbf{s.t.} & x_1 - x_2 + 5x_3 \leq 5 \\ & 2x_1 - x_2 + 3x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

From the optimal tableau, show that all the alternative optima are not corner points (i.e., nonbasic).

5. For the following LP, show that the optimal solution is degenerate and that none of the alternative solutions are corner points.

$$\begin{array}{ll}
 \mathbf{max} & z = 3x_1 + x_2 \\
 \mathbf{s.t.} & x_1 + 2x_2 \leq 5 \\
 & x_1 + x_2 - x_3 \leq 2 \\
 & 7x_1 + 3x_2 - 5x_3 \leq 20 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

6. Consider the LP:

$$\begin{array}{ll}
 \mathbf{max} & z = 20x_1 + 5x_2 + x_3 \\
 \mathbf{s.t.} & 3x_1 + 5x_2 - 5x_3 \leq 50 \\
 & x_1 \leq 10 \\
 & x_1 + 3x_2 - 4x_3 \leq 20 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

- (a) By inspecting the constraints, determine the direction  $(x_1, x_2, x_3)$  in which the solution space is unbounded.
- (b) Without further computations, what can you conclude regarding the optimum objective value?

7. Consider the LP model

$$\begin{array}{ll}
 \mathbf{max} & z = 3x_1 + 2x_2 + 3x_3 \\
 \mathbf{s.t.} & 2x_1 + x_2 + x_3 \leq 4 \\
 & 3x_1 + 4x_2 + 2x_3 \geq 16 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

Use hand computations to show that the optimal solution can include an artificial basic variable at zero level. Does the problem have a feasible optimal solution?

8. The following tableau represents a specific simplex iteration. All variables are nonnegative. The tableau is not optimal for either maximization or minimization. Thus, when a nonbasic variable enters the solution, it can either increase or decrease  $z$  or leave it unchanged, depending on the parameters of the entering nonbasic variable.

Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	RHS
$z$	0	-5	0	4	-1	-10	0	0	620
$x_8$	0	3	0	-2	-3	-1	5	1	12
$x_3$	0	1	1	3	1	0	3	0	6
$x_1$	1	-1	0	0	6	-4	0	0	0

- (a) Categorize the variables as basic and nonbasic, and provide the current values of all the variables.
- (b) Assuming that the problem is of the maximization type, identify the nonbasic variables that have the potential to improve the value of  $z$ . If each such variable enters the basic solution, determine the associated leaving variable, if any, and the associated change in  $z$ .
- (c) Repeat part (b) assuming that the problem is of the minimization type.
- (d) Which nonbasic variable(s) will not cause a change in the value of  $z$  when selected to enter the solution?

9. You are given the tableau shown below for a maximization problem.

Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
$z$	$-c$	2	0	0	0	10
$x_3$	-1	$a_1$	1	0	0	4
$x_4$	$a_2$	-4	0	1	0	1
$x_5$	$a_3$	3	0	0	1	$b$

Give conditions on the unknowns  $a_1$ ,  $a_2$ ,  $a_3$ ,  $b$ , and  $c$  that make the following statements true:

- (a) The current solution is optimal.
  - (b) The current solution is optimal, and there are alternative optimal solutions.
  - (c) The LP is unbounded (in this part, assume that  $b \geq 0$ ).
10. Suppose we have obtained the tableau shown below for a maximization problem.

Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$z$	$c_1$	$c_2$	0	0	0	0	10
$x_3$	4	$a_1$	1	0	$a_2$	0	$b$
$x_4$	-1	-5	0	1	-1	0	2
$x_6$	$a_3$	-3	0	0	-4	1	3

State conditions on  $a_1$ ,  $a_2$ ,  $a_3$ ,  $b$ ,  $c_1$ , and  $c_2$  that are required to make the following statements true:

- (a) The current solution is optimal, and there are alternative optimal solutions.
- (b) The current basic solution is not a basic feasible solution.
- (c) The current basic solution is a degenerate bfs.

- (d) The current basic solution is feasible, but the LP is unbounded.
- (e) The current basic solution is feasible, but the objective function value can be improved by replacing  $x_6$  as a basic variable with  $x_1$ .
11. The starting and current tableaux of a given problem are shown below. Find the values of the unknowns  $a$  through  $n$ .

<b>Starting Tableau</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
	$z$	$a$	1	-3	0	0	0
	$x_4$	$b$	$c$	$d$	1	0	6
	$x_5$	-1	2	$e$	0	1	1
<b>Current Tableau</b>	Basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
	$z$	0	$-1/3$	$j$	$k$	$\ell$	$n$
	$x_1$	$g$	$2/3$	$2/3$	$2/3$	0	$f$
	$x_5$	$h$	$i$	$-1/3$	$2/3$	1	$m$

### 3 Sensitivity Analysis and Duality

Two of the most important topics in linear programming are sensitivity analysis and duality. After studying these important topics, the reader will have an appreciation of the beauty and logic of linear programming.

#### 3.1 Some Important Formulas

In this section, we use our knowledge of matrices to show how an LP's optimal tableau can be expressed in terms of the LP's parameters. The formulas developed in this section are used in our study of sensitivity analysis and duality.

Assume that we are solving a max problem that has  $m$  constraints and  $n$  variables. Although some of these variables may be slack, excess, or artificial, we choose to label them  $x_1, x_2, \dots, x_n$ . Then the LP may be written as

$$\begin{aligned}
 \text{max} \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
 \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
 & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
 & x_1, x_2, \dots, x_n \geq 0
 \end{aligned} \tag{3.1}$$

Suppose we have found the optimal solution to (3.1). We define:

##### Definition 3.1.

1.  $BV$  is the set of basic variables in optimal tableau.
2.  $\mathbf{X}_{BV}$  is the  $m \times 1$  vector of basic variables in optimal tableau.
3.  $NBV$  is the set of nonbasic variables in optimal tableau.
4.  $\mathbf{X}_{NBV}$  is the  $(n - m) \times 1$  vector of nonbasic variables in optimal tableau.
5.  $\mathbf{C}_{BV}$  is the  $1 \times m$  row vector contains the coefficients of basic variables in the initial tableau.
6.  $\mathbf{C}_{NBV}$  is the  $1 \times (n - m)$  row vector contains the coefficients of nonbasic variables in the initial tableau.
7. The  $m \times m$  matrix  $\mathbf{B}$  is the matrix whose  $j$ th column is the column of  $BV_j$  in the initial tableau.

8.  $\mathbf{a}_j$  is the column (in the constraints) of the variable  $x_j$  in the initial tableau.
9.  $\mathbf{N}$  is the  $m \times (n-m)$  matrix whose columns are the coefficients of the nonbasic variables in the initial tableau.
10.  $\mathbf{b}$  is the  $1 \times m$  column vector contains the right-hand side of the constraints in the initial tableau.

So, the LP in (3.1) can be written as

$$\begin{aligned} \max \quad & z = \mathbf{C}_{BV} \mathbf{X}_{BV} + \mathbf{C}_{NBV} \mathbf{X}_{NBV} \\ \text{s.t.} \quad & \mathbf{B} \mathbf{X}_{BV} + \mathbf{N} \mathbf{X}_{NBV} = \mathbf{b} \\ & \mathbf{X}_{BV}, \mathbf{X}_{NBV} \geq \mathbf{0} \end{aligned} \quad (3.2)$$

and the initial tableau has the form

Basic	$BV$	$NBV$	RHS
$z$	$\mathbf{C}_{BV}$	$\mathbf{C}_{NBV}$	$z$ value
$\mathbf{X}_{BV}$	$\mathbf{B}_{m \times m}$	$\mathbf{N}_{m \times (n-m)}$	$\mathbf{b}$

### Formulas for Computing the Optimal Tableau

1. Right-hand side of optimal tableau's constraints:  $\bar{\mathbf{b}} = \mathbf{B}^{-1} \mathbf{b}$ .
2. Right-hand side of optimal row 0:  $\bar{z} = \mathbf{C}_{BV} (\mathbf{B}^{-1} \mathbf{b}) = \mathbf{C}_{BV} \bar{\mathbf{b}}$ .
3.  $x_j$  columns in optimal tableau's constraints:  $\bar{\mathbf{a}}_j = \mathbf{B}^{-1} \mathbf{a}_j$ . In general,  $\mathbf{X}_{NBV}$  columns in optimal tableau's constraints:  $\bar{\mathbf{N}} = \mathbf{B}^{-1} \mathbf{N}$  and  $\mathbf{X}_{NBV}$  column in optimal tableau's constraints =  $\mathbf{I}_m$ .
4. (a)  $\bar{c}_j = \mathbf{C}_{BV} (\mathbf{B}^{-1} \mathbf{a}_j) - c_j = \mathbf{C}_{BV} \bar{\mathbf{a}}_j - c_j$ .  
 (b)  $\bar{c}_{s_i} = i$ th element in  $\mathbf{C}_{BV} \mathbf{B}^{-1}$ .  
 (c)  $\bar{c}_{e_i} = -i$ th element in  $\mathbf{C}_{BV} \mathbf{B}^{-1}$ .  
 (d) In maximization LP,  $\bar{c}_{a_i} = i$ th element in  $\mathbf{C}_{BV} \mathbf{B}^{-1} + M$ .  
 (e) In minimization LP,  $\bar{c}_{a_i} = i$ th element in  $\mathbf{C}_{BV} \mathbf{B}^{-1} - M$ .  
 (f) **In general**,  $\bar{\mathbf{C}}_{NBV} = \mathbf{C}_{BV} (\mathbf{B}^{-1} \mathbf{N}) - \mathbf{C}_{NBV} = \mathbf{C}_{BV} \bar{\mathbf{N}} - \mathbf{C}_{NBV}$ .

The optimal tableau then has the form:

Basic	$BV$	$NBV$	RHS
$z$	$\mathbf{0}$	$\mathbf{C}_{BV} (\mathbf{B}^{-1} \mathbf{N}) - \mathbf{C}_{NBV}$	$\mathbf{C}_{BV} (\mathbf{B}^{-1} \mathbf{b})$
$\mathbf{X}_{BV}$	$\mathbf{I}_{m \times m}$	$\mathbf{B}^{-1} \mathbf{N}_{m \times (n-m)}$	$\mathbf{B}^{-1} \mathbf{b}$

### Formulas Derivation:

1. To expressing the constraints in any tableau in terms of  $\mathbf{B}^{-1}$  and the original LP, we observe that

$$\mathbf{B}\mathbf{X}_{BV} + \mathbf{N}\mathbf{X}_{NBV} = \mathbf{b}$$

Multiplying both sides from the left by  $\mathbf{B}^{-1}$ , we obtain

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{X}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$

which implies that

$$\mathbf{X}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$

2. To determining the optimal tableau's Row 0 in terms of the initial LP, we rewrite the original objective function,  $z = \mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{NBV}\mathbf{X}_{NBV}$  as

$$z - \mathbf{C}_{BV}\mathbf{X}_{BV} - \mathbf{C}_{NBV}\mathbf{X}_{NBV} = 0 \quad (3.3)$$

Also, we multiply the constraints expressed in the form  $\mathbf{B}\mathbf{X}_{BV} + \mathbf{N}\mathbf{X}_{NBV} = \mathbf{b}$  through by the vector  $\mathbf{C}_{BV}\mathbf{B}^{-1}$  to obtain

$$\mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{b} \quad (3.4)$$

By adding equation (3.3) to equation (3.4), we obtain

$$z + (\mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV})\mathbf{X}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{b}$$

**Example 3.1.** For the following LP, the optimal basis is  $BV = \{x_2, s_2\}$ . Compute the optimal tableau.

$$\begin{array}{ll} \mathbf{max} & z = x_1 + 4x_2 \\ \mathbf{s.t.} & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

**Solution:** After adding slack variables  $s_1$  and  $s_2$ , the LP in standard form

$$\begin{array}{ll} \mathbf{max} & z = x_1 + 4x_2 \\ \mathbf{s.t.} & x_1 + 2x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

Since  $BV = \{x_2, s_2\}$  and  $NBV = \{x_1, s_1\}$ , then

$$\begin{array}{lll} \mathbf{C}_{BV} = [4 & 0] & , \quad \mathbf{C}_{NBV} = [1 \quad 0] \quad , \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \\ \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} & , \quad \mathbf{B}^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} & , \quad \mathbf{N} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \end{array}$$



So, the optimal tableau entries are

$$\begin{aligned}\bar{\mathbf{b}} &= \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ \bar{\mathbf{N}} &= \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} \\ \bar{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\bar{\mathbf{N}} - \mathbf{C}_{NBV} = [4 \ 0] \begin{bmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} - [1 \ 0] = [1 \ 2] \\ \bar{z} &= \mathbf{C}_{BV}\bar{\mathbf{b}} = [4 \ 0] \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 12.\end{aligned}$$

and the optimal tableau is

Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
$z$	1	0	2	0	12
$x_2$	$1/2$	1	$1/2$	0	3
$s_2$	$3/2$	0	$-1/2$	1	5

**Example 3.2.** For the following LP, the optimal basis is  $BV = \{x_2, x_4\}$ . Compute the optimal tableau.

$$\begin{aligned}\mathbf{max} \quad & z = x_1 + 4x_2 + 7x_3 + 5x_4 \\ \mathbf{s.t.} \quad & 2x_1 + x_2 + 2x_3 + 4x_4 = 10 \\ & 3x_1 - x_2 - 2x_3 + 6x_4 = 5 \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

**Solution:** Note that the constraints are in equation form, and no need to add artificial variables here (we do not solve by simplex). Since  $BV = \{x_2, x_4\}$  and  $NBV = \{x_1, x_3\}$ , then

$$\begin{aligned}\mathbf{C}_{BV} &= [4 \ 5] \quad , \quad \mathbf{C}_{NBV} = [1 \ 7] \quad , \quad \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} \quad , \quad \mathbf{N} = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix}\end{aligned}$$

So, the optimal tableau entries are

$$\begin{aligned}\bar{\mathbf{b}} &= \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3/2 \end{bmatrix} \\ \bar{\mathbf{N}} &= \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix} \\ \bar{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\bar{\mathbf{N}} - \mathbf{C}_{NBV} = [4 \ 5] \begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix} - [1 \ 7] = [3/2 \ 1]\end{aligned}$$

$$\bar{z} = \mathbf{C}_{BV} \bar{\mathbf{b}} = \begin{bmatrix} 4 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3/2 \end{bmatrix} = \frac{47}{2}.$$

and the optimal tableau is

Basic	$x_1$	$x_2$	$x_3$	$x_4$	RHS
$z$	$3/2$	0	1	0	$47/2$
$x_2$	0	1	2	0	4
$x_4$	$1/2$	0	0	1	$3/2$

**Note 11.** We have used the formulas of this section to create an LP's optimal tableau, but they can also be used to create the tableau for any set of basic variables.

### Exercise 3.1.

1. For the following LP,  $x_1$  and  $x_2$  are basic variables in the optimal tableau. Use the formulas of matrices to determine the optimal tableau.

$$\begin{array}{ll} \mathbf{max} & z = 3x_1 + x_2 \\ \mathbf{s.t.} & 2x_1 - x_2 \leq 2 \\ & -x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

2. For the following LP,  $x_2$  and  $s_1$  are basic variables in the optimal tableau. Use the formulas of matrices to determine the optimal tableau.

$$\begin{array}{ll} \mathbf{max} & z = -x_1 + x_2 \\ \mathbf{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

3. Consider the following LP model:

$$\begin{array}{ll} \mathbf{max} & z = 5x_1 + 2x_2 + 3x_3 \\ \mathbf{s.t.} & x_1 + 5x_2 + 2x_3 \leq b_1 \\ & x_1 - 5x_2 - 6x_3 \leq b_2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The following optimal tableau corresponds to specific values of  $b_1$  and  $b_2$ :

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS
$z$	0	$a$	7	$d$	$e$	150
$x_1$	1	$b$	2	1	0	30
$s_2$	0	$c$	-8	-1	1	10

Determine the elements  $a, b, c, d, e, b_1$  and  $b_2$ .

4. For the following LP,  $x_1$  and  $x_2$  are basic variables in the optimal tableau. Determine the optimal tableau using the laws of matrices.

$$\begin{array}{ll}\min & w = 50x_1 + 100x_2 \\ \text{s.t.} & 7x_1 + 2x_2 \geq 28 \\ & 2x_1 + 12x_2 \geq 24 \\ & x_1, x_2 \geq 0\end{array}$$

### 3.2 Sensitivity Analysis

We now explore how changes in an LP's parameters (objective function coefficients, right-hand sides, and technological coefficients) change the optimal solution. The study of how an LP's optimal solution depends on its parameters is called sensitivity analysis. Our discussion focuses on maximization problems and relies heavily on the formulas of Section 3.1. (The modifications for min problems are straightforward; see Exercise 3.2 at the end of this section.)

As in Section 3.1, we let  $BV$  be the set of basic variables in the optimal tableau. Given a change (or changes) in an LP, we want to determine whether  $BV$  remains optimal. The mechanics of sensitivity analysis hinge on the following important observation. From Chapter 2, we know that a simplex tableau (for a max problem) for a set of basic variables  $BV$  is optimal if and only if each constraint has a nonnegative right-hand side and each variable has a nonnegative coefficient in row 0. This implies that whether a tableau is feasible and optimal depends only on the right-hand sides of the constraints and on the coefficients of each variable in row 0.

Suppose we have solved an LP and have found that  $BV$  is an optimal basis. We can use the following procedure to determine if any change in the LP will cause  $BV$  to be no longer optimal.

**Step 1:** Using the formulas of Section 3.1, determine how changes in the LP's parameters change the right-hand side and row 0 of the optimal tableau (the tableau having  $BV$  as the set of basic variables).

**Step 2:** If each variable in row 0 has a non-negative coefficient and each constraint has a nonnegative right-hand side, then  $BV$  is still optimal. Otherwise,  $BV$  is no longer optimal.

We will discuss how 6 types of changes on LP's parameters change the optimal solution:

1. **Changing the objective function coefficient of a nonbasic variable:** If the objective function coefficient for a nonbasic variable  $x_j$  is changed, the current basis remains optimal if  $\bar{c}_j \geq 0$ . If  $\bar{c}_j < 0$ , then the current basis is no longer optimal, and  $x_j$  will be a basic variable in the new optimal solution.

2. **Changing the objective function coefficient of a basic variable:**

If the objective function coefficient of a basic variable  $x_j$  is changed, then the current basis remains optimal if the coefficient of every variable in row 0 of the  $BV$  tableau remains nonnegative. If any variable in row 0 has a negative coefficient, then the current basis is no longer optimal.

3. **Changing the right-hand side of a constraint:**

If the right-hand side of a constraint is changed, then the current basis remains optimal if the right-hand side of each constraint in the tableau remains nonnegative. If the right-hand side of any constraint is negative, then the current basis is infeasible, and a new optimal solution must be found.

4. **Changing a column of a nonbasic variable:**

If the column of a nonbasic variable  $x_j$  is changed, then the current basis remains optimal if  $\bar{c}_j \geq 0$ . If  $\bar{c}_j < 0$ , then the current basis is no longer optimal and  $x_j$  will be a basic variable in the new optimal solution. If the column of a basic variable is changed, then it is usually difficult to determine whether the current basis remains optimal. This is because the change may affect both  $\mathbf{B}$  and  $\mathbf{C}_{BV}$  and thus the entire row 0 and the entire right-hand side of the optimal tableau. As always, the current basis would remain optimal if and only if each variable has a nonnegative coefficient in row 0 and each constraint has a nonnegative right-hand side.

5. **Adding a new variable:** If a new column (corresponding to a variable  $x_j$ ) is added to an LP, then the current basis remains optimal if  $\bar{c}_j \geq 0$ . If  $\bar{c}_j < 0$ , then the current basis is no longer optimal and  $x_j$  will be a basic variable in the new optimal solution.

6. **Adding a new constraint.** (see Section 3.8)

The following figure presents a summary of sensitivity analyses for a maximization problem. When applying the techniques of this section to a mini-

Summary of Sensitivity Analysis (Max Problem)

Change in Initial Problem	Effect on Optimal Tableau	Current Basis Is Still Optimal If:
Changing nonbasic objective function coefficient $c_j$	Coefficient of $x_j$ in optimal row 0 is changed	Coefficient of $x_j$ in row 0 for current basis is still nonnegative
Changing basic objective function coefficient $c_j$	Entire row 0 may change	Each variable still has a nonnegative coefficient in row 0
Changing right-hand side of a constraint	Right-hand side of constraints and row 0 are changed	Right-hand side of each constraint is still nonnegative
Changing the column of a nonbasic variable $x_j$ or adding a new variable $x_j$	Changes the coefficient for $x_j$ in row 0 and $x_j$ 's constraint column in optimal tableau	The coefficient of $x_j$ in row 0 is still nonnegative

mization problem, just remember that a tableau is optimal if and only if each variable has a nonpositive coefficient in row 0 and the right-hand side of each constraint is nonnegative.

**Example 3.3.** Consider the following LP:

$$\begin{aligned} \max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + \frac{3}{2}x_3 \leq 20 \\ & 2x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

After adding slack variables  $s_1, s_2$ , and  $s_3$ , the optimal tableau is:

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	5	0	0	10	10	280
$s_1$	0	-2	0	1	2	-8	24
$x_3$	0	-2	1	0	2	-4	8
$x_1$	1	$5/4$	0	0	$-1/2$	$3/2$	2

1. Suppose we change the objective function coefficient of  $x_2$  from 30 to  $30 + \Delta$ . For what values of  $\Delta$  will the current set of basic variables remain optimal ?

**Solution:** From the optimal tableau we know that  $BV = \{s_1, x_3, x_1\}$  and  $NBV = \{x_2, s_2, s_3\}$ , then

$$\begin{aligned} \mathbf{C}_{BV} &= [0 \quad 20 \quad 60] \quad , \quad \mathbf{C}_{NBV} = [30 \quad 0 \quad 0] \quad , \quad \mathbf{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 1 & 1 & 8 \\ 0 & 3/2 & 4 \\ 0 & 1/2 & 2 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \quad , \quad \mathbf{N} = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} \end{aligned}$$

Because  $x_2$  is a nonbasic variable,  $\mathbf{C}_{BV}$  has not changed. Thus,  $BV$  will remain optimal if

$$\begin{aligned} \bar{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= [0 \quad 20 \quad 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} - [30 + \Delta \quad 0 \quad 0] \\ &= [5 - \Delta \quad 10 \quad 10] \geq \mathbf{0} \\ \therefore \quad \Delta &\leq 5 \end{aligned}$$

2. Suppose we change the objective function coefficient of  $x_1$  from 60 to  $60 + \Delta$ . For what values of  $\Delta$  will the current set of basic variables remain optimal ?

**Solution:** The  $BV$  will remain optimal if

$$\begin{aligned}\bar{C}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= [0 \quad 20 \quad 60 + \Delta] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} - [30 \quad 0 \quad 0] \\ &= [5 + \frac{5}{4}\Delta \quad 10 - \frac{1}{2}\Delta \quad 10 + \frac{3}{2}\Delta] \geq \mathbf{0} \\ \therefore \Delta &\in [-4, 20]\end{aligned}$$

3. Suppose we change the right-hand-side of the second constraint from 20 to  $20 + \Delta$ . For what values of  $\Delta$  will the current set of basic variables remain optimal ?

**Solution:** The  $BV$  will remain optimal if

$$\begin{aligned}\bar{\mathbf{b}} &= \mathbf{B}^{-1}\mathbf{b} \\ &= \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 48 \\ 20 + \Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24 + 2\Delta \\ 8 + 2\Delta \\ 2 - \frac{1}{2}\Delta \end{bmatrix} \geq \mathbf{0} \\ \therefore \Delta &\in [-4, 4]\end{aligned}$$

4. Suppose we change the elements of the column for  $x_2$  from  $\begin{bmatrix} 30 \\ 6 \\ 2 \\ 3/2 \end{bmatrix}$  to  $\begin{bmatrix} 43 \\ 5 \\ 2 \\ 2 \end{bmatrix}$ . Would this change the optimal solution to the problem ?

**Solution:** Thus,  $BV$  will remain optimal if  $\bar{C}_{NBV} \geq \mathbf{0}$ . But

$$\begin{aligned}\bar{C}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= [0 \quad 20 \quad 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} - [43 \quad 0 \quad 0] \\ &= [-3 \quad 10 \quad 10] \not\geq \mathbf{0} \\ \therefore \text{The current basis is no longer optimal.}\end{aligned}$$

5. Suppose we add new activity  $x_4$  to the problem, and we add the column

$$\begin{bmatrix} 15 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

for  $x_4$  to the problem. How will the addition of the new activity change the optimal tableau?

**Solution:** Because

$$\begin{aligned} \bar{c}_{x_4} &= \mathbf{C}_{BV} \mathbf{B}^{-1} \mathbf{a}_4 - c_{x_4} \\ &= [0 \quad 20 \quad 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 \\ &= 5 \geq 0 \end{aligned}$$

$\therefore$  The current basis is still optimal.

**Exercise 3.2.** Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and men. To reach these groups, Dorian Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercial spots on two types of programs: comedy shows and football games. Each comedy commercial is seen by 7 million high-income women and 2 million high-income men. Each football commercial is seen by 2 million high-income women and 12 million high-income men. A 1-minute comedy ad costs \$50,000, and a 1-minute football ad costs \$100,000. Dorian would like the commercials to be seen by at least 28 million high-income women and 24 million high-income men. Suppose

$x_1$  = number of 1-minute comedy ads purchased,

$x_2$  = number of 1-minute football ads purchased.

Then the formulation of this model is

$$\begin{aligned} \min \quad & w = 50x_1 + 100x_2 \\ \text{s.t.} \quad & 7x_1 + 2x_2 \geq 28 \quad (\text{HIW}) \\ & 2x_1 + 12x_2 \geq 24 \quad (\text{HIM}) \\ & x_1, x_2 \geq 0 \end{aligned}$$

The optimal tableau is given in the table below. Remember that for a min problem, a tableau is optimal if and only if each variable has a nonpositive coefficient in row 0 and the right-hand side of each constraint is nonnegative.

Basic	$x_1$	$x_2$	$e_1$	$e_2$	$a_1$	$a_2$	RHS
$z$	0	0	-5	-15/2	$5 - M$	$15/2 - M$	320
$x_1$	1	0	-3/20	1/40	3/20	-1/40	18/5
$x_2$	0	1	1/40	-7/80	-1/40	7/80	7/5

1. Find the range of values of the cost of a comedy ad (currently \$50,000) for which the current basis remains optimal.
2. Find the range of values of the number of required HIW exposures (currently 28 million) for which the current basis remains optimal. If 40 million HIW exposures were required, what would be the new optimal solution?
3. Suppose an ad on a news program costs \$110,000 and reaches 12 million HIW and 7 million HIM. Should Dorian advertise on the news program?

### 3.3 Finding the Dual of an LP

Associated with any LP is another LP, called the *dual*. Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear and nonlinear programming. This relation is important because it gives us interesting economic insights. Knowledge of duality will also provide additional insights into sensitivity analysis.

When taking the dual of a given LP, we refer to the given LP as the *primal*. The two problems are closely related, in the sense that the optimal solution of one problem automatically provides the optimal solution to the other. If the primal is a max problem, then the dual will be a min problem, and vice versa. We begin by explaining how to find the dual of a max problem in which all variables are required to be nonnegative and all constraints are ( $\leq$ ) constraints (called a *normal max problem*). A normal max problem may be written as

$$\begin{array}{ll}
 \mathbf{max} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
 \mathbf{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
 & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
 & x_1, x_2, \cdots, x_n \geq 0
 \end{array} \tag{3.5}$$

The dual of a normal max problem such as (3.5) is defined to be

$$\begin{array}{ll}
 \mathbf{min} & w = b_1y_1 + b_2y_2 + \cdots + b_my_m \\
 \mathbf{s.t.} & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
 & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
 & \vdots \\
 & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
 & y_1, y_2, \cdots, y_m \geq 0
 \end{array} \tag{3.6}$$

A min problem such as (3.6) that has all ( $\geq$ ) constraints and all variables nonnegative is called a *normal min problem*. If the primal is a normal min problem such as (3.6), then we define the dual of (3.6) to be (3.5).



### Finding the Dual:

1. If the primal is a **maximization** problem, the dual will be a **minimization** problem, and vice versa. The dual of the dual problem yields the original problem.
2. A *dual variable* is defined for each *primal constraint* equation. Also, a *dual constraint* is defined for each *primal variable*.
3. The column coefficients in a constraint of a primal variable defines the left-hand side coefficients of the dual constraint and its objective coefficient defines the right-hand side of that constraint. The objective coefficients of the dual equal the right-hand side of the primal constraint.
4. The variables and constraints in the primal and dual problems are related as follows.

(max) Constraint Sign	$\Longleftrightarrow$	(min) Variable Sign
$\geq$	$\Longleftrightarrow$	$\leq 0$
$\leq$	$\Longleftrightarrow$	$\geq 0$
$=$	$\Longleftrightarrow$	u.r.s
Variable Sign	$\Longleftrightarrow$	Constraint Sign
$\leq 0$	$\Longleftrightarrow$	$\leq$
$\geq 0$	$\Longleftrightarrow$	$\geq$
u.r.s	$\Longleftrightarrow$	$=$

**Example 3.4.** Find the dual of the following LPs:

- |  |   |
|--|---|
| <p>1. <b>min</b>     <math>w = x_1 + 2x_2</math><br/> <b>s.t.</b>     <math>2x_1 + x_2 \geq 4</math><br/> <math>x_1 - 2x_2 \geq 8</math><br/> <math>x_1, x_2 \geq 0</math></p>                         | <p>3. <b>min</b>     <math>w = 15x_1 + 12x_2</math><br/> <b>s.t.</b>     <math>x_1 + 2x_2 \geq 3</math><br/> <math>x_1 - 4x_2 \leq 5</math><br/> <math>x_1, x_2 \geq 0</math></p>   |
| <p>2. <b>max</b>     <math>z = 5x_1 + 12x_2 + 4x_3</math><br/> <b>s.t.</b>     <math>x_1 + 2x_2 + x_3 \leq 10</math><br/> <math>2x_1 - x_2 + x_3 = 8</math><br/> <math>x_1, x_2, x_3 \geq 0</math></p> | <p>4. <b>max</b>     <math>z = 5x_1 + 6x_2</math><br/> <b>s.t.</b>     <math>x_1 + 2x_2 = 5</math><br/> <math>-x_1 + 5x_2 \geq 3</math><br/> <math>4x_1 + 7x_2 \leq 8</math><br/> <math>x_1</math> urs, <math>x_2 \geq 0</math></p> |

**Solution:**

- |   |  |
|---|--|
| <p>1. <b>max</b>     <math>z = 4y_1 + 8y_2</math><br/> <b>s.t.</b>     <math>2y_1 + y_2 \leq 1</math><br/> <math>y_1 - 2y_2 \leq 2</math><br/> <math>y_1, y_2 \geq 0</math></p> | <p>2. <b>min</b>     <math>w = 10y_1 + 8y_2</math><br/> <b>s.t.</b>     <math>y_1 + 2y_2 \geq 5</math><br/> <math>2y_1 - y_2 \geq 12</math><br/> <math>y_1 + y_2 \geq 4</math><br/> <math>y_1 \geq 0, y_2</math> urs</p> |
|---|--|

- |   |  |
|---|--|
| 3. <b>max</b> $z = 3y_1 + 5y_2$<br><b>s.t.</b> $y_1 + y_2 \leq 15$<br>$2y_1 - 4y_2 \leq 12$<br>$y_1 \geq 0, y_2 \leq 0$ | 4. <b>min</b> $w = 5y_1 + 3y_2 + 8y_3$<br><b>s.t.</b> $y_1 - y_2 + 4y_3 = 5$<br>$2y_1 + 5y_2 + 7y_3 \geq 6$<br>$y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0$ |
|---|--|

**Exercise 3.3.** Find the dual of the following LPs:

- |   |  |
|---|--|
| 1. <b>max</b> $z = 2x_1 + x_2$<br><b>s.t.</b> $-x_1 + x_2 \leq 1$<br>$x_1 + x_2 \leq 3$<br>$x_1 - 2x_2 \leq 4$<br>$x_1, x_2 \geq 0$ | 3. <b>max</b> $z = 4x_1 - x_2 + 2x_3$<br><b>s.t.</b> $x_1 + x_2 \leq 5$<br>$2x_1 + x_2 \leq 7$<br>$2x_2 + x_3 \geq 6$<br>$x_1 + x_3 = 4$<br>$x_1 \geq 0, x_2, x_3 \text{ urs}$ |
| 2. <b>min</b> $w = y_1 - y_2$<br><b>s.t.</b> $2y_1 + y_2 \geq 4$<br>$y_1 + y_2 \geq 1$<br>$y_1 + 2y_2 \geq 3$<br>$y_1, y_2 \geq 0$  | 4. <b>min</b> $w = 4y_1 + 2y_2 - y_3$<br><b>s.t.</b> $y_1 + 2y_2 \leq 6$<br>$y_1 - y_2 + 2y_3 = 8$<br>$y_1, y_2 \geq 0, y_3 \text{ urs}$                                       |

### 3.4 The Dual Theorem and its Consequences

In this section, we discuss one of the most important results in linear programming: the Dual Theorem. In essence, the Dual Theorem states that the primal and dual have equal optimal objective function values (if the problems have optimal solutions).

If we choose any feasible solution to the max LP and any feasible solution to the min LP (one is primal and the other is dual), the value for the min LP feasible solution will be at least as large as the value for the max LP feasible solution. This result is formally stated in Lemma 3.1. Observe that the following two results say nothing about which problem is primal and which is dual. The objective function type, max or min, that matters in this case.

**Lemma 3.1.** *The objective values in a pair of primal-dual problems must satisfy the following relationships:*

1. *For any pair of feasible primal and dual solutions,*

$$\left( \begin{array}{c} \text{objective value} \\ \text{in MAX LP} \end{array} \right) \leq \left( \begin{array}{c} \text{objective value} \\ \text{in MIN LP} \end{array} \right)$$

2. *At the optimum solution for both problems,*

$$\left( \begin{array}{c} \text{objective value} \\ \text{in MAX LP} \end{array} \right) = \left( \begin{array}{c} \text{objective value} \\ \text{in MIN LP} \end{array} \right)$$

*Proof.* Consider the primal LP

$$\begin{array}{ll} \max & z = \mathbf{C}_{BV} \mathbf{X}_{BV} + \mathbf{C}_{NBV} \mathbf{X}_{NBV} \\ \text{s.t.} & \mathbf{A} \mathbf{X}_{NBV} + \mathbf{I} \mathbf{X}_{BV} = \mathbf{b} \\ & \mathbf{X}_{BV}, \mathbf{X}_{NBV} \geq \mathbf{0} \end{array} \quad (3.7)$$

Then, the dual LP will be

$$\begin{array}{ll} \min & w = \mathbf{Y} \mathbf{b} \\ \text{s.t.} & \mathbf{Y} \mathbf{A} \geq \mathbf{C}_{NBV} \\ & \mathbf{Y} \mathbf{I} \geq \mathbf{C}_{BV} \\ & \mathbf{Y} \text{ free} \end{array} \quad (3.8)$$

Multiply the constraint in (3.7) by  $\mathbf{Y}$  from the left to obtain:

$$\mathbf{Y} \mathbf{A} \mathbf{X}_{NBV} + \mathbf{Y} \mathbf{I} \mathbf{X}_{BV} = \mathbf{Y} \mathbf{b} = w.$$

Also, multiply the first constraint in (3.8) by  $\mathbf{X}_{BV}$  from the right, and the second constraint by  $\mathbf{X}_{NBV}$  from the right, to obtain:

$$\begin{array}{l} \mathbf{Y} \mathbf{A} \mathbf{X}_{BV} \geq \mathbf{C}_{NBV} \mathbf{X}_{BV} \\ \mathbf{Y} \mathbf{I} \mathbf{X}_{NBV} \geq \mathbf{C}_{BV} \mathbf{X}_{NBV} \end{array}$$

By adding the two inequalities above, we have

$$\begin{array}{l} \mathbf{Y} \mathbf{A} \mathbf{X}_{NBV} + \mathbf{Y} \mathbf{I} \mathbf{X}_{BV} \geq \mathbf{C}_{BV} \mathbf{X}_{BV} + \mathbf{C}_{NBV} \mathbf{X}_{NBV} \\ \underbrace{w}_{\min} \geq \underbrace{z}_{\max} \end{array}$$

■

**Example 3.5.** Consider the following pair of primal and dual problems.

Primal	Dual
$\begin{array}{ll} \min & w = 5x_1 + 2x_2 \\ \text{s.t.} & x_1 - x_2 \geq 3 \\ & 2x_1 + 3x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{array}$	$\begin{array}{ll} \max & z = 3y_1 + 5y_2 \\ \text{s.t.} & y_1 + 2y_2 \leq 5 \\ & -y_1 + 3y_2 \leq 2 \\ & y_1, y_2 \geq 0 \end{array}$
<b>Feasible Solution:</b> $x_1 = 4, x_2 = 1$	<b>Feasible Solution (Optimal):</b> $y_1 = 5, y_2 = 0$
<b>Objective Function:</b> $w = 22$	<b>Objective Function:</b> $z = 15$

**Theorem 3.2. The Dual Theorem.<sup>a</sup>**

Suppose  $BV$  is an optimal basis for the primal. Then  $\mathbf{y} = \mathbf{C}_{BV}\mathbf{B}^{-1}$  is an optimal solution to the dual. Also,  $\bar{z} = \bar{w}$ .

<sup>a</sup>For proof see: Wayne L. Winston, Munirpallam Venkataramanan. **Introduction to Mathematical Programming**. Thomson Learning; 4th edition (2002)

**Example 3.6.** The optimal solution of the following LP is  $z = 9$  when  $x_1 = 1$  and  $x_2 = 6$ . Find its dual problem, then find the solution for the dual problem.

$$\begin{array}{ll}\mathbf{max} & z = 3x_1 + x_2 \\ \mathbf{s.t.} & 2x_1 + x_2 \leq 8 \\ & 4x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0\end{array}$$

**Solution:** Since, in the optimal solution,  $BV = \{x_1, x_2\}$  then

$$\mathbf{C}_{BV} = [3 \ 1] \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 2 & -1 \end{bmatrix}$$

Hence,  $\bar{\mathbf{y}} = [y_1 \ y_2] = \mathbf{C}_{BV}\mathbf{B}^{-1} = [1/2 \ 1/2]$  and  $\bar{w} = \bar{z} = 9$ . Note that the dual LP is

$$\begin{array}{ll}\mathbf{min} & w = 8y_1 + 10y_2 \\ \mathbf{s.t.} & 2y_1 + 4y_2 \geq 3 \\ & y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0\end{array}$$

**How to Read the Optimal Dual Solution from Row 0 of the Optimal Tableau ?**

Constraint $i$ Sign	Optimal $y_i$ Value	Problem Type
$\leq$	Coefficient of $s_i$	Max or Min
$\geq$	$-1 \times$ Coefficient of $e_i$	Max or Min
$=$	Coefficient of $a_i - M$	Max
$=$	Coefficient of $a_i + M$	Min

In general,

$$\left\{ \begin{array}{l} \text{The optimal} \\ \text{value of the dual} \\ \text{variable } y_i \end{array} \right\} = \left\{ \begin{array}{l} \text{Optimal primal } z - \text{coefficient of the starting variable } d_i \\ + \\ \text{Original objective function coefficient of } d_i \end{array} \right\}$$

**Example 3.7.** Consider the following LP.

$$\begin{array}{ll}\mathbf{max} & z = -2x_1 - x_2 + x_3 \\ \mathbf{s.t.} & x_1 + x_2 + x_3 \leq 3 \\ & x_2 + x_3 \geq 2 \\ & x_1 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

1. Find the dual of this LP.

**Solution:** The dual LP is

$$\begin{array}{ll}\mathbf{min} & w = 3y_1 + 2y_2 + y_3 \\ \mathbf{s.t.} & y_1 + y_3 \geq -2 \\ & y_1 + y_2 \geq -1 \\ & y_1 + y_2 + y_3 \geq 1 \\ & y_1 \geq 0, y_2 \leq 0, y_3 \text{ urs}\end{array}$$

2. After adding slack variable  $s_1$ , subtracting excess variable  $e_2$ , and adding artificial variables  $a_2$  and  $a_3$ , the **Row 0** of the LP's optimal tableau is found to be

$$z + 4x_1 + e_2 + (M - 1)a_2 + (M + 2)a_3 = 0.$$

Find the optimal solution of the dual problem.

**Solution:** The starting primal variables  $s_1$ ,  $a_2$  and  $a_3$  uniquely correspond to the dual variables  $y_1$ ,  $y_2$  and  $y_3$ , respectively. Thus, the optimum dual solution is  $\bar{w} = \bar{z} = 0$ , and

Using the Table	Using the General Formula
$y_1 = 0$	$y_1 = 0 + 0 = 0$
$y_2 = -1$	$y_2 = -M + (M - 1) = -1$
$y_3 = (M + 2) - M = 2$	$y_3 = -M + (M + 2) = 2$

3. Suppose we change the right-hand side of the third constraint from 1 to 2, what is the change on the  $z$ -value will be? Assume the current basis remains optimal.

$$\mathbf{Solution: } z_{\text{new}} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\mathbf{y}} \mathbf{b}_{\text{new}} = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = 2.$$

4. Repeat part (3) if we change the right-hand side of the second constraint from 2 to 5.

$$\mathbf{Solution: } z_{\text{new}} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\mathbf{y}} \mathbf{b}_{\text{new}} = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = -3.$$

**Corollary 3.3.** *The primal problem is infeasible if and only if the normal form of the dual problem is unbounded (and vice versa).*

**Note 12.** With regard to the primal and dual linear programming problems, exactly one of the following statements is true:

1. Both possess optimal solutions.
2. One problem has an unbounded optimal objective value, in which case the other problem must be infeasible.
3. Both problems are infeasible.

From this note we see that duality is not completely symmetric. The best we can say is that (here optimal means having a finite optimum, and unbounded means having an unbounded optimal objective value):

Primal <b>Optimal</b>	$\Leftrightarrow$	Dual <b>Optimal</b>
Primal (Dual) <b>Unbounded</b>	$\Rightarrow$	Dual (Primal) <b>Infeasible</b>
Primal (Dual) <b>Infeasible</b>	$\Rightarrow$	Dual (Primal) <b>Unbounded or Infeasible</b>
Primal (Dual) <b>Infeasible</b>	$\Leftrightarrow$	Dual (Primal) <b>Unbounded in normal form</b>

**Note 13.** The relationship between degeneracy and multiplicity of the primal and the dual optimal solutions is formulated in Theorem 3.4. Recall that *degeneracy* and *multiplicity* always refer to LP models with inequality constraints, and that *degeneracy* is defined for basic feasible solutions. In this theorem, the term *nondegenerate* in the expression “multiple and nondegenerate” means that there are multiple optimal solutions, and that there exists an optimal basic feasible solution that is nondegenerate.

**Theorem 3.4. Duality relationships between degeneracy and multiplicity.**

*For any pair of primal and dual standard LP-models where both have optimal solutions, the following implications hold:*

Primal optimal solution		Dual optimal solution
<i>Multiple</i>	$\Rightarrow$	<i>Degenerate</i>
<i>Unique and nondeg.</i>	$\Rightarrow$	<i>Unique and nondeg.</i>
<i>Multiple and nondeg.</i>	$\Rightarrow$	<i>Unique and degenerate</i>
<i>Unique and degenerate</i>	$\Rightarrow$	<i>Multiple</i>

**Exercise 3.4.**

- Find the optimal value of the objective function for the following LP using its dual. (Do NOT solve the dual using the simplex algorithm)

$$\begin{array}{ll}\min & w = 10y_1 + 4y_2 + 5y_3 \\ \text{s.t.} & 5y_1 - 7y_2 + 3y_3 \geq 50 \\ & y_1, y_2, y_3 \geq 0\end{array}$$

- Consider the following LP.

$$\begin{array}{ll}\max & z = 2x_1 + 4x_2 + 4x_3 - 3x_4 \\ \text{s.t.} & x_1 + x_2 + x_3 = 4 \\ & 4x_2 + x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0\end{array}$$

- Write the associated dual problem.
- Show that the basic solution  $x_1$  and  $x_2$  is not optimal.
- Using  $x_3$  and  $x_4$  as starting variables, the optimal tableau is given below. Determine the dual optimal solution in **TWO** ways, using the tableau.

Basic	$x_1$	$x_2$	$x_3$	$x_4$	RHS
$z$	2	0	0	3	16
$x_3$	$3/4$	0	1	$-1/4$	2
$x_2$	$1/4$	1	0	$1/4$	2

- For the following LP,

$$\begin{array}{ll}\max & z = -x_1 + 5x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 0.5 \\ & -x_1 + 3x_2 \leq 0.5 \\ & x_1, x_2 \geq 0\end{array}$$

row 0 of the optimal tableau is  $z + 0.4s_1 + 1.4s_2 = \boxed{?}$ . Determine the optimal  $z$ -value for the given LP.

- Consider the following linear programming problem:

$$\begin{array}{ll}\max & z = 4x_1 + x_2 \\ \text{s.t.} & 3x_1 + 2x_2 \leq 6 \\ & 6x_1 + 3x_2 \leq 10 \\ & x_1, x_2 \geq 0\end{array}$$

Suppose that in solving this problem, row 0 of the optimal tableau is found to be  $z + 2x_2 + s_2 = \frac{20}{3}$ . Use the Dual Theorem to prove that the computations must be incorrect.

5. Solve the dual of the following problem, and then find its optimal solution from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$\begin{aligned}
 \text{min} \quad & w = 50x_1 + 60x_2 + 30x_3 \\
 \text{s.t.} \quad & 5x_1 + 5x_2 + 3x_3 \geq 50 \\
 & x_1 + x_2 - x_3 \geq 20 \\
 & 7x_1 + 6x_2 - 9x_3 \geq 30 \\
 & 5x_1 + 5x_2 + 5x_3 \geq 35 \\
 & 2x_1 + 4x_2 - 15x_3 \geq 10 \\
 & 12x_1 + 10x_2 \geq 90 \\
 & x_2 - 10x_3 \geq 20 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

6. Consider the following LP:

$$\begin{aligned}
 \text{max} \quad & z = 5x_1 + 2x_2 + 3x_3 \\
 \text{s.t.} \quad & x_1 + 5x_2 + 2x_3 = 15 \\
 & x_1 - 5x_2 - 6x_3 \leq 20 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Given that the artificial variable  $a_1$  and the slack variable  $s_2$  form the starting basic variables and that  $M$  was set equal to 100 when solving the problem, the optimal tableau is given as:

Basic	$x_1$	$x_2$	$x_3$	$a_1$	$s_2$	RHS
$z$	0	23	7	105	0	75
$x_1$	1	5	2	1	0	15
$s_2$	0	-10	-8	-1	1	5

Write the associated dual problem, and determine its optimal solution in two ways.



### 3.5 Shadow Prices

It is often important for managers to determine how a change in a constraint's right-hand side changes the LP's optimal  $z$ -value.

**Definition 3.2.** The shadow price of the  $i$ th constraint is the amount by which the optimal value of the objective function is improved (improved means increased in max LP and decreased in min LP) if we increase  $b_i$  (the RHS of that constraint) by 1 (from  $b_i$  to  $b_i + 1$ ).

**Note 14.**

1. The previous definition assumes that after the RHS of constraint  $i$  has been changed to  $b_i + 1$ , the current basis remains optimal.
2. The shadow price of the  $i$ th constraint of a max LP is the optimal value of the  $i$ th dual variable  $\bar{y}_i$ . Also, the shadow price of the  $i$ th constraint of a min LP is  $-1 \times$  the optimal value of the  $i$ th dual variable  $\bar{y}_i$ .

$$\begin{aligned}
 \bar{z}_{\text{new}} &= \mathbf{Y}\mathbf{b} = [\bar{y}_1 \quad \cdots \quad \bar{y}_i \quad \cdots \quad \bar{y}_m] \begin{bmatrix} b_1 \\ \vdots \\ b_i + 1 \\ \vdots \\ b_m \end{bmatrix} \\
 &= \bar{y}_1 b_1 + \cdots + \bar{y}_i (b_i + 1) + \cdots + \bar{y}_m b_m \\
 &= (\bar{y}_1 b_1 + \cdots + \bar{y}_i b_i + \cdots + \bar{y}_m b_m) + \bar{y}_i \\
 &= \bar{z}_{\text{old}} + \bar{y}_i \\
 \therefore \bar{z}_{\text{new}} - \bar{z}_{\text{old}} &= \bar{y}_i
 \end{aligned}$$

3. In max LP, the shadow price for a  $(\leq)$  constraint is nonnegative, for a  $(\geq)$  is nonpositive, and for  $(=)$  is urs. Also, in min LP, the shadow price for a  $(\leq)$  constraint is nonpositive, for a  $(\geq)$  is nonnegative, and for  $(=)$  is urs.
4. In general, if the RHS of the  $i$ th constraint is increased by an amount  $\Delta b_i$ , then

$$\begin{aligned}
 \bar{z}_{\text{new}} &= \bar{z}_{\text{old}} + (\Delta b_i) \bar{y}_i && \text{if the LP is max} \\
 \bar{z}_{\text{new}} &= \bar{z}_{\text{old}} - (\Delta b_i) \bar{y}_i && \text{if the LP is min}
 \end{aligned}$$

**Example 3.8.** Consider the following LP.

$$\begin{aligned}
 \max \quad & z = 15x_1 + 25x_2 \\
 \text{s.t.} \quad & 3x_1 + 4x_2 \leq 100 \\
 & 2x_1 + 3x_2 \leq 70 \\
 & x_1 + 2x_2 \leq 30 \\
 & x_2 \geq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

The optimal solution of the problem is  $z = 435$ , when  $x_1 = 24$  and  $x_2 = 3$ , where the Row 0 in the optimal tableau (after adding slack variables  $s_1, s_2, s_3$  to the first three constraints respectively and subtracting excess variable  $e_4$  from the last constraint then adding to it an artificial variable  $a_4$ ) is

$$z + 15s_3 + 5e_4 + (M - 5)a_4 = 435.$$

1. Find the shadow price of each constraint.

**Solution:** The shadow price of each constraint is the optimal value of the corresponding dual variable of each constraint. So,

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = 15, \quad y_4 = -5$$

2. Assuming the current basis remains optimal, what would the change on the  $z$ -value be if the RHS of the

- (a) 3rd constraint were changed from 30 to 35 ?

$$\text{Solution: } \bar{z}_{\text{new}} = \bar{\mathbf{y}}\mathbf{b} = [0 \quad 0 \quad 15 \quad -5] \begin{bmatrix} 100 \\ 70 \\ 35 \\ 3 \end{bmatrix} = 510.$$

- (b) 4th constraint were changed from 3 to 2 ?

$$\text{Solution: } \bar{z}_{\text{new}} = \bar{\mathbf{y}}\mathbf{b} = [0 \quad 0 \quad 15 \quad -5] \begin{bmatrix} 100 \\ 70 \\ 30 \\ 2 \end{bmatrix} = 440.$$

**Exercise 3.5.** Sugarco can manufacture three types of candy bar. Each candy bar consists totally of sugar and chocolate. The compositions of each type of candy bar and the profit earned from each candy bar are shown in the table below.

Bar	Amount of Sugar (Ounces)	Amount of Chocolate (Ounces)	Profit (Cents)
1	1	2	3
2	1	3	7
3	1	1	5

Fifty oz of sugar and 100 oz of chocolate are available. After defining  $x_i$  to be the number of Type  $i$  candy bars manufactured, Sugarco should solve the following LP:

$$\begin{aligned} \max \quad & z = 3x_1 + 7x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 50 \\ & 2x_1 + 3x_2 + x_3 \leq 100 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

After adding slack variables  $s_1$  and  $s_2$ , the optimal tableau is as shown in the table below.

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS
$z$	3	0	0	4	1	300
$x_3$	$1/2$	0	1	$3/2$	$-1/2$	25
$x_2$	$1/2$	1	0	$-1/2$	$1/2$	25

Using this optimal tableau, answer the following questions:

1. Find the shadow prices for the Sugarco problem.
2. If 60 oz of sugar were available, what would be Sugarco's profit?

### 3.6 Duality and Sensitivity Analysis

From the dual theorem we demonstrate the following: Assuming that a set of basic variables  $BV$  is feasible, then  $BV$  is optimal if and only if the associated dual solution  $\mathbf{C}_{BV}\mathbf{B}^{-1}$  is dual feasible.

$$\text{Unbounded} \implies \text{infeasible, then feasible} \implies \text{bounded}$$

This result can be used for an alternative way of doing the following types of sensitivity analysis:

1. Changing the objective function coefficient of a nonbasic variable.
2. Changing a column of a nonbasic variable.
3. Adding a new variable.

Since primal optimality and dual feasibility are equivalent, the above changes will leave the current basic optimal if and only if the current dual solution  $\mathbf{C}_{BV}\mathbf{B}^{-1}$  remains dual feasible.

**Example 3.9.** Consider the following LP.

$$\begin{aligned} \max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \leq 48 \end{aligned}$$

$$\begin{aligned}
4x_1 + 2x_2 + \frac{3}{2}x_3 &\leq 20 \\
2x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 &\leq 8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

The dual of the problem is:

$$\begin{aligned}
\min \quad & w = 48y_1 + 20y_2 + 8y_3 \\
\text{s.t.} \quad & 8y_1 + 4y_2 + 2y_3 \geq 60 \\
& 6y_1 + 2y_2 + \frac{3}{2}y_3 \geq 30 \\
& y_1 + \frac{3}{2}y_2 + \frac{1}{2}y_3 \geq 20 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

The optimal solution for the primal was  $z = 280$ ,  $\underbrace{s_1 = 24, x_3 = 8, x_1 = 2}_{\text{Basic}}$ ,  $\underbrace{x_2 = s_2 = s_3 = 0}_{\text{NonBasic}}$ . Also, the optimal dual solution (constraint shadow prices) are  $y_1 = 0, y_2 = 10, y_3 = 10$ .

1. Let  $c_2$  be the coefficient of  $x_2$  in the objective function. For what values of  $c_2$  will the current basis remain optimal?

**Solution:** If  $y_1 = 0, y_2 = 10, y_3 = 10$  remains dual feasible, then the current basis and the values of all the variables are unchanged. Note that if the objective function coefficient for  $x_2$  is changed, then the first and third dual constraints remain unchanged, but the second dual constraint is changed to  $6y_1 + 2y_2 + \frac{3}{2}y_3 \geq c_2$ . Thus, the current basis remains optimal if  $c_2$  satisfies  $6(0) + 2(10) + \frac{3}{2}(10) \geq c_2$ , or  $c_2 \leq 35$ .

2. Suppose we change the elements of the column for  $x_2$  from  $\begin{bmatrix} 30 \\ 6 \\ 2 \end{bmatrix}$  to

$$\begin{bmatrix} 43 \\ 5 \\ 2 \\ 2 \end{bmatrix}, \text{ does the current basis remain optimal?}$$

**Solution:** Changing the column for the nonbasic variable leaves the first and third dual constraints unchanged but changes the second to

$$5y_1 + 2y_2 + 2y_3 \geq 43$$

Because  $y_1 = 0, y_2 = 10, y_3 = 10$  does not satisfy the new second dual constraint, dual feasibility is not maintained, and the current basis is no longer optimal.

3. Suppose we add new activity  $x_4$  to the problem, and we add of the  $x_4$  column  $\begin{bmatrix} 15 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  to the problem. Does the current basis remain optimal?

**Solution:** Introducing the new activity leaves the three dual constraints unchanged, but the new variable  $x_4$  adds a new dual constraint. The new dual constraint will be  $y_1 + y_2 + y_3 \geq 15$ . Because  $0 + 10 + 10 \geq 15$ , the current basis remains optimal.

### Exercise 3.6.

- The following questions refer to the Sugarco problem (Exercise 3.5):
  - For what values of profit on a Type 1 candy bar does the current basis remain optimal?
  - If a Type 1 candy bar used 0.5 oz of sugar and 0.75 oz of chocolate, would the current basis remain optimal?
  - A Type 4 candy bar is under consideration. A Type 4 candy bar yields a 10¢ profit and uses 2 oz of sugar and 1 oz of chocolate. Does the current basis remain optimal?
- Consider the following LP and its optimal tableau:

$$\begin{aligned}
 \max \quad & z = 5x_1 + x_2 + 2x_3 \\
 \text{s.t.} \quad & x_1 + x_2 + 1x_3 \leq 6 \\
 & 6x_1 + 1x_3 \leq 8 \\
 & x_2 + 1x_3 \leq 2 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	$1/6$	0	0	$5/6$	$1/6$	9
$s_1$	0	$1/6$	0	1	$-1/6$	$-5/6$	3
$x_1$	1	$-1/6$	0	0	$1/6$	$-1/6$	1
$x_3$	0	1	1	0	0	1	2

- Find the dual to this LP and its optimal solution.
- Find the range of values of the objective function coefficient of  $x_2$  for which the current basis remains optimal.
- Find the range of values of the objective function coefficient of  $x_1$  for which the current basis remains optimal.

### 3.7 Complementary Slackness

The Theorem of Complementary Slackness is an important result that relates the optimal primal and dual solutions. To state this theorem, we assume that the primal is a normal max problem with variables  $x_1, x_2, \dots, x_n$  and  $m$  of ( $\leq$ ) constraints. Then the dual is a normal min problem with variables  $y_1, y_2, \dots, y_m$  and  $n$  of ( $\geq$ ) constraints.

$$\begin{array}{ll}
 \mathbf{max} & z = c_1x_1 + \dots + c_nx_n \\
 \mathbf{s.t.} & a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\
 & a_{21}x_1 + \dots + a_{2n}x_n \leq b_2 \\
 & \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 & a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \\
 & x_i \geq 0, \forall i = 1, 2, \dots, n
 \end{array}
 \qquad
 \begin{array}{ll}
 \mathbf{min} & w = b_1y_1 + \dots + b_my_m \\
 \mathbf{s.t.} & a_{11}y_1 + \dots + a_{m1}y_m \geq c_1 \\
 & a_{12}y_1 + \dots + a_{m2}y_m \geq c_2 \\
 & \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 & a_{1n}y_1 + \dots + a_{mn}y_m \geq c_n \\
 & y_j \geq 0, \forall j = 1, 2, \dots, m
 \end{array}$$

Let  $s_1, s_2, \dots, s_m$  be the slack variables for the primal, and  $e_1, e_2, \dots, e_n$  be the excess variables for the dual.

$$\begin{array}{ll}
 \mathbf{max} & z = c_1x_1 + \dots + c_nx_n \\
 \mathbf{s.t.} & a_{11}x_1 + \dots + a_{1n}x_n + s_1 = b_1 \\
 & a_{21}x_1 + \dots + a_{2n}x_n + s_2 = b_2 \\
 & \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 & a_{m1}x_1 + \dots + a_{mn}x_n + s_m = b_m \\
 & x_i \geq 0, \forall i = 1, 2, \dots, n \\
 & s_j \geq 0, \forall j = 1, 2, \dots, m
 \end{array}
 \qquad
 \begin{array}{ll}
 \mathbf{min} & w = b_1y_1 + \dots + b_my_m \\
 \mathbf{s.t.} & a_{11}y_1 + \dots + a_{m1}y_m - e_1 = c_1 \\
 & a_{12}y_1 + \dots + a_{m2}y_m - e_2 = c_2 \\
 & \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 & a_{1n}y_1 + \dots + a_{mn}y_m - e_n = c_n \\
 & y_j \geq 0, \forall j = 1, 2, \dots, m \\
 & e_i \geq 0, \forall i = 1, 2, \dots, n
 \end{array}$$

**Theorem 3.5.** Let  $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  be a feasible primal solution

and  $\bar{y} = [y_1 \ \dots \ y_m]$  be a feasible dual solution. Then  $\bar{x}$  is primal optimal and  $\bar{y}$  is dual optimal, ( $\bar{z} = \bar{w}$ ), if and only if

$$\begin{aligned}
 x_i e_i &= 0, \quad \forall i = 1, 2, \dots, n \\
 y_j s_j &= 0, \quad \forall j = 1, 2, \dots, m
 \end{aligned}$$

*In other words, if a constraint in either the primal or dual is non binding ( $s_j > 0$  or  $e_i > 0$ ), then the corresponding (complementary) variable in the other problem must equal 0.*

*Proof.* Multiply each constraint in the primal in standard form by its corresponding (complementary) dual variable:

$$\begin{array}{rcl} a_{11}x_1y_1 + \cdots + a_{1n}x_ny_1 + s_1y_1 & = & b_1y_1 \\ a_{21}x_1y_2 + \cdots + a_{2n}x_ny_2 + s_2y_2 & = & b_2y_2 \\ \vdots & & \vdots \\ a_{m1}x_1y_m + \cdots + a_{mn}x_ny_m + s_my_m & = & b_my_m \end{array}$$

Add all the constraints above together:

$$\begin{aligned}
& (a_{11}x_1y_1 + \cdots + a_{1n}x_ny_1 + \cdots \\
& \quad + a_{m1}x_1y_m + \cdots + a_{mn}x_ny_m) \\
& + (s_1y_1 + \cdots + s_my_m) \\
& = b_1y_1 + \cdots + b_my_m \\
& = w
\end{aligned} \tag{3.9}$$

Now, multiply each constraint in the dual in standard form by its corresponding (complementary) primal variable:

$$\begin{array}{rcl} a_{11}y_1x_1 + \cdots + a_{m1}y_mx_1 - e_1x_1 & = & c_1x_1 \\ a_{12}y_1x_2 + \cdots + a_{m2}y_mx_2 - e_2x_2 & = & c_2x_2 \\ \vdots & & \vdots \\ a_{1n}y_1x_n + \cdots + a_{mn}y_mx_n - e_nx_n & = & c_nx_n \end{array}$$

Add all the constraints above together:

$$\begin{aligned}
& (a_{11}y_1x_1 + \cdots + a_{m1}y_mx_1 + \cdots \\
& \quad + a_{1n}y_1x_n + \cdots + a_{mn}y_mx_n) \\
& - (e_1x_1 + \cdots + e_nx_n) \\
& = c_1x_1 + \cdots + c_nx_n \\
& = z
\end{aligned} \tag{3.10}$$

Subtract equation (3.10) from equation (3.9) to obtain:

$$s_1y_1 + s_2y_2 + \cdots + s_my_m + e_1x_1 + e_2x_2 + \cdots + e_nx_n = w - z \quad (3.11)$$

If  $\bar{x}$  is primal optimal and  $\bar{y}$  is dual optimal, then  $z = w$ , and then

$$s_1y_1 + s_2y_2 + \cdots + s_my_m + e_1x_1 + e_2x_2 + \cdots + e_nx_n = 0$$

Since, all  $x$ 's,  $y$ 's,  $s$ 's, and  $e$ 's are all nonnegative, then

$$\begin{aligned} x_i e_i &= 0, \quad \forall i = 1, 2, \dots, n \\ y_j s_j &= 0, \quad \forall j = 1, 2, \dots, m \end{aligned}$$

Also, if

$$\begin{aligned} x_i e_i &= 0, \quad \forall i = 1, 2, \dots, n \\ y_j s_j &= 0, \quad \forall j = 1, 2, \dots, m \end{aligned}$$

then  $0 = z - w$  and hence  $z = w$ . So,  $\bar{x}$  is primal optimal and  $\bar{y}$  is dual optimal. ■

**Example 3.10.** Consider the following LP.

$$\begin{aligned} \text{max} \quad & z = 5x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq 6 \\ & x_1 + 2x_2 + x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The optimal solution to the problem is  $z = \frac{49}{3}$ ,  $x_1 = \frac{5}{3}$ ,  $x_2 = \frac{8}{3}$ , and  $x_3 = s_1 = s_2 = 0$ . Use complementary slackness theorem to find the optimal dual solution of the problem.

**Solution:** The dual LP is

$$\begin{aligned} \text{min} \quad & w = 6y_1 + 7y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \geq 5 \\ & y_1 + 2y_2 \geq 3 \\ & y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Because  $x_1 > 0$  and  $x_2 > 0$  then the optimal dual solution must have  $e_1 = 0$  and  $e_2 = 0$ . This means that for the optimal dual solution, the first and second constraints must be binding. So we know that the optimal values of  $y_1$  and  $y_2$  may be found by solving the first and second dual constraints as equalities. Thus, the optimal values of  $y_1$  and  $y_2$  must satisfy

$$\begin{aligned} 2y_1 + y_2 &= 5 \\ y_1 + 2y_2 &= 3 \end{aligned}$$

Solving these equations simultaneously shows that the optimal dual solution must have  $y_1 = \frac{7}{3}$  and  $y_2 = \frac{1}{3}$ , with  $\bar{w} = \bar{z} = \frac{49}{3}$ .



**Exercise 3.7.** Consider the following LP.

$$\begin{aligned} \mathbf{max} \quad & z = 3x_1 + 4x_2 + x_3 + 5x_4 \\ \mathbf{s.t.} \quad & x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \\ & 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The optimal solution to the problem is  $z = 13$ ,  $x_1 = 1$ ,  $x_2 = x_3 = 0$ , and  $x_4 = 2$ . Use complementary slackness theorem to find the optimal dual solution of the problem.

### 3.8 The Dual–Simplex Method

In the *simplex algorithm* presented in Chapter 2 the problem starts at a (basic) feasible solution. Successive iterations continue to be feasible until the optimal is reached at the last iteration. The algorithm is sometimes referred to as the primal simplex method.

This section presents two additional algorithms: The *dual simplex* and the *generalized simplex*.

- In the *dual simplex*, the LP starts at a better than optimal infeasible (basic) solution. Successive iterations remain infeasible and (better than) optimal until feasibility is restored at the last iteration.
- The *generalized simplex* combines both the primal and dual simplex methods in one algorithm. It deals with problems that start both non-optimal and infeasible. In this algorithm, successive iterations are associated with basic feasible or infeasible (basic) solutions. At the final iteration, the solution becomes optimal and feasible (assuming that one exists).

**Dual Simplex Algorithm:** The crux of the dual simplex method is to start with a better than optimal and infeasible basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while moving the solution iterations toward feasibility.

- To start the LP optimal and infeasible, two requirements must be met:
  - The objective function must satisfy the optimality condition of the regular simplex method.
  - All the constraints must be of the type ( $\leq$ ), regardless the type of problem either max or min. This condition requires converting any ( $\geq$ ) to ( $\leq$ ) simply by multiplying both sides of the inequality ( $\geq$ ) by  $-1$ . If the LP includes ( $=$ ) constraints, the equation can be replaced by two inequalities. For example,  $x_1 + x_2 = 1$  is equivalent to  $\begin{cases} x_1 + x_2 \leq 1 \\ x_1 + x_2 \geq 1 \end{cases}$  or  $\begin{cases} x_1 + x_2 \leq 1 \\ -x_1 - x_2 \leq -1 \end{cases}$

- After converting all the constraints to  $(\leq)$ , the starting solution is infeasible if at least one of the right-hand sides of the inequalities is strictly negative.
- **Dual feasibility condition.** The leaving variable,  $x_r$  is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.
- **Dual optimality condition.** Given that  $x_r$  is the leaving variable, let  $\bar{c}_j$  be the reduced cost of nonbasic variable  $x_j$  and  $a_{rj}$  the constraint coefficient in the  $x_r$ -row and  $x_j$ -column of the tableau. The entering variable is the nonbasic variable with  $a_{rj} < 0$  that corresponds to

$$\min_{\text{NonBasic } x_j} \left\{ \left| \frac{\bar{c}_j}{a_{rj}} \right|, a_{rj} < 0 \right\}$$

Note that

1. Ties are broken arbitrarily.
2. If  $a_{rj} \geq 0$  for all nonbasic  $x_j$ , the problem has no feasible solution.

**Example 3.11.** Use the dual-simplex algorithm for solving the following LP problem.

$$\begin{aligned} \min \quad & w = 5x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 4x_1 - x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

**Solution:** After converting all the constraints to  $(\leq)$ , then adding slack variables  $s_1$  and  $s_2$  to the constraints, the LP in standard form is

$$\begin{aligned} \min \quad & w - 5x_1 - 6x_2 = 0 \\ \text{s.t.} \quad & -x_1 - x_2 + s_1 = -2 \\ & -4x_1 + x_2 + s_2 = -4 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

The initial tableau and all following tableaus, using the dual-simplex algorithm, are shown below.

		↓					
Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	<b>Optimal but not feasible</b>
	$w$	-5	-6	0	0	0	
	$s_1$	-1	-1	1	0	-2	
	$s_2$	-4	1	0	1	-4	
		←					

Iteration [1]					↓		Optimal but not feasible
	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	
	$w$	0	$-29/4$	0	$-5/4$	5	
	← $s_1$	0	$-5/4$	1	$-1/4$	-1	
	$x_1$	1	$-1/4$	0	$-1/4$	1	

Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS	Optimal and feasible
		0	-1	-5	0		
		0	5	-4	1	4	
		1	1	-1	0	2	

**Example 3.12.** Use the dual-simplex algorithm for solving the following LP problem.

$$\begin{aligned}
 \max \quad & z = -4x_1 - 2x_2 \\
 \text{s.t.} \quad & x_1 + x_2 = 1 \\
 & -3x_1 + x_2 \geq 2 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

**Solution:** After converting all the constraints to ( $\leq$ ), then adding slack variables  $s_1, s_2$  and  $s_3$  to the constraints, the LP in standard form is

$$\begin{aligned}
 \max \quad & z + 4x_1 + 2x_2 = 0 \\
 \text{s.t.} \quad & x_1 + x_2 + s_1 = 1 \\
 & -x_1 - x_2 + s_2 = -1 \\
 & 3x_1 - x_2 + s_3 = -2 \\
 & x_1, x_2, s_1, s_2, s_3 \geq 0
 \end{aligned}$$

The initial tableau and all following tableaus, using the dual-simplex algorithm, are shown below.

Iteration [0]	Basic	<div>↓</div>					RHS	Optimal but not feasible		
		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$				
		$z$	4	2	0	0			0	
		$s_1$	1	1	1	0			0	1
		$s_2$	-1	-1	0	1			0	-1
		← $s_3$	3	-1	0	0			1	-2

Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS	Since $s_1$ leaves with no entering variable, then the solution is infeasible
	$z$	10	0	0	0	2	-4	
←	$s_1$	4	0	1	0	1	-1	
	$s_2$	-4	0	0	1	-1	1	
	$x_2$	-3	1	0	0	-1	2	

**Note 15.** The dual simplex method is often used to find the new optimal solution to an LP after a constraint is added. When a constraint is added, one of the following three cases will occur:

1. The current optimal solution satisfies the new constraint.
2. The current optimal solution does not satisfy the new constraint, but the LP still has a feasible solution.
3. The additional constraint causes the LP to have no feasible solutions.

**Example 3.13.** Consider the following LP and its optimal tableau.

<b>max</b>	$z = 6x_1 + x_2$						
<b>s.t.</b>	$x_1 + x_2 \leq 5$						
	$2x_1 + x_2 \leq 6$						
	$x_1, x_2 \geq 0$						
		Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
		$z$	0	2	0	3	18
		$s_1$	0	1/2	1	-1/2	2
		$x_1$	1	1/2	0	1/2	3

Find the optimal solution to this LP if we add the constraint

1.  $3x_1 + x_2 \leq 10$ .

**Solution:** After converting the constraint to  $(=)$  by adding  $s_3$  to the constraint, we obtain:

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	2	0	3	0	18
$s_1$	0	1/2	1	-1/2	0	2
$x_1$	1	1/2	0	1/2	0	3
$s_3$	3	1	0	0	1	10

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	2	0	3	0	18
$s_1$	0	1/2	1	-1/2	0	2
$x_1$	1	1/2	0	1/2	0	3
$s_3$	0	-1/2	0	-3/2	1	1

The solution is optimal and feasible

2.  $x_1 - 2x_2 \geq 6$ .

**Solution:** After converting the constraint to  $(\leq)$  then adding  $s_3$  to the constraint, we obtain:

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	2	0	3	0	18
$s_1$	0	1/2	1	-1/2	0	2
$x_1$	1	1/2	0	1/2	0	3
$s_3$	-1	2	0	0	1	-6

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS	Since $s_3$ leaves with no entering variable, then the solution is infeasible
$z$	0	2	0	3	0	18	
$s_1$	0	$1/2$	1	$-1/2$	0	2	
$x_1$	1	$1/2$	0	$1/2$	0	3	
$s_3$	0	$5/2$	0	$1/2$	1	-3	

3.  $8x_1 + x_2 \leq 12$ .

**Solution:** After converting the constraint to ( $=$ ) by adding  $s_3$  to the constraint, we obtain:

Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
	$z$	0	2	0	3	0	18
	$s_1$	0	$1/2$	1	$-1/2$	0	2
	$x_1$	1	$1/2$	0	$1/2$	0	3
	$s_3$	8	1	0	0	1	12

↓

Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS	The solution is optimal but infeasible
	$z$	0	2	0	3	0	18	
	$s_1$	0	$1/2$	1	$-1/2$	0	2	
	$x_1$	1	$1/2$	0	$1/2$	0	3	
←	$s_3$	0	-3	0	-4	1	-12	

Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS	The solution is optimal and feasible
	$z$	0	0	0	$1/3$	$2/3$	10	
	$s_1$	0	0	1	$-7/3$	$1/6$	0	
	$x_1$	1	0	0	$-1/6$	$1/6$	1	
	$x_2$	0	1	0	$4/3$	$-1/3$	4	

**Generalized Simplex Algorithm:** The (primal) simplex algorithm starts feasible but not optimal. The dual simplex starts (better than) optimal but infeasible. What if an LP model starts both not optimal and infeasible? The following example illustrates what we call the generalized simplex algorithm for solving LP problems with this situation.

**Example 3.14.** Consider the following LP.

$$\begin{aligned}
 \max \quad & z = x_1 - 3x_2 \\
 \text{s.t.} \quad & x_1 - x_2 \leq 2 \\
 & x_1 + x_2 \geq 4 \\
 & 2x_1 + 2x_2 \geq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

The model can be put in the following tableau form in which the starting basic solution  $(s_1, s_2, s_3)$  is both non-optimal (because  $x_1$  has a negative reduced cost) and infeasible (because  $s_2 = -4$ ,  $s_3 = -3$ ).

		↓						<b>The solution is not optimal and infeasible</b>
←	Iteration [0]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
		$z$	-1	3	0	0	0	0
		$s_1$	1	-1	1	0	0	2
		$s_2$	-1	-1	0	1	0	-4
		$s_3$	-2	-2	0	0	1	-3

Remove infeasibility first by applying a version of the dual simplex feasibility condition that selects  $s_2$  as the leaving variable. To determine the entering variable, all we need is a nonbasic variable whose constraint coefficient in the  $s_2$ -row is strictly negative. The selection can be done without regard to optimality, because it is nonexistent at this point anyway. In the present example,  $x_1$  and  $x_2$  have negative coefficient in the  $s_2$ -row and  $x_1$  is selected as the entering variable. The result is the following tableau:

		↓						<b>The solution is not optimal and infeasible</b>
←	Iteration [1]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
		$z$	0	4	0	-1	0	4
		$s_1$	0	-2	1	1	0	-2
		$x_1$	1	1	0	-1	0	4
		$s_3$	0	0	0	-2	1	5

At this point,  $s_1$  leaves the solution and  $x_2$  have negative coefficient in the  $s_1$ -row and is selected as the entering variable. The result is the following tableau:

								<b>The solution is optimal and feasible</b>
	Iteration [2]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
		$z$	0	0	2	1	0	0
		$x_2$	0	1	-1/2	-1/2	0	1
		$x_1$	1	0	1/2	-1/2	0	3
		$s_3$	0	0	0	-2	1	5

The solution in the preceding tableau is now feasible and fortunately optimal.

**Dual Simplex with Artificial Constraints:** In example (3.14), the dual simplex is not applicable directly, because  $x_1$  does not satisfy the maximization optimality condition. Show that by adding the artificial constraint  $x_1 \leq M$  (where  $M$  is sufficiently large not to eliminate any feasible points in the original solution space), and then using the new constraint as a pivot row, the selection of  $x_1$  as the entering variable (because it has the most negative objective coefficient) will render an all-optimal objective row. Next, carry out the regular dual simplex method on the modified problem. All the iterations are shown below.

Iteration [0]	Basic	↓						RHS
		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
		$z$	-1	3	0	0	0	
		$s_1$	1	-1	1	0	0	
		$s_2$	-1	-1	0	1	0	
		$s_3$	-2	-2	0	0	1	
←	$s_4$	↓						$M$
		1	0	0	0	0	1	

Iteration [1]	Basic	↓						RHS
		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
		$z$	0	3	0	0	0	
		$s_1$	0	-1	1	0	0	
		$s_2$	0	-1	0	1	0	
		$s_3$	0	-2	0	0	1	
←	$x_1$	↓						$M$
		1	0	0	0	0	1	

Iteration [2]	Basic	↓						RHS
		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
		$z$	0	2	1	0	0	
		$s_4$	0	1	-1	0	0	
		$s_2$	0	-2	1	1	0	
		$s_3$	0	-4	2	0	1	
←	$x_1$	↓						2
		1	-1	1	0	0	0	

Iteration [3]	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	RHS
		$z$	0	0	2	1	0	
		$s_4$	0	0	-1/2	1/2	0	
		$x_2$	0	1	-1/2	-1/2	0	
		$s_3$	0	0	0	-2	1	
		$x_1$	1	0	1/2	-1/2	0	

The solution  
is optimal  
and feasible

### Exercise 3.8.

1. Solve the following LP

$$\begin{aligned}
 \max \quad & z = 2x_1 - x_2 + x_3 \\
 \text{s.t.} \quad & 2x_1 + 4x_2 - x_3 \geq 4 \\
 & -x_1 + x_2 - x_3 \geq 2 \\
 & x_1 + 2x_2 + 2x_3 \leq 8 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

by using

- (a) the generalized simplex algorithm,
- (b) the dual simplex algorithm after adding the constraint  $x_1 + x_3 \leq M$ .

2. Use the dual simplex method to solve the following LP:

$$\begin{aligned}
 \mathbf{max} \quad & z = -2x_1 - x_3 \\
 \mathbf{s.t.} \quad & x_1 + x_2 - x_3 \geq 5 \\
 & x_1 - 2x_2 + 4x_3 \geq 8 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

3. Solve the following LP in three different ways. Which method appears to be the most efficient computationally?

$$\begin{aligned}
 \mathbf{min} \quad & w = 6x_1 + 7x_2 + 3x_3 + 5x_4 \\
 \mathbf{s.t.} \quad & 5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12 \\
 & x_2 - 5x_3 - 6x_4 \geq 10 \\
 & 2x_1 + 5x_2 + x_3 + x_4 \geq 8 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

4. Instead of solving a problem using all of its constraints, we can start by identifying the so called secondary constraints. These are the constraints that we suspect are least restrictive in terms of the optimum solution. The model is solved using the remaining (primary) constraints. We may then add the secondary constraints one at a time. A secondary constraint is discarded if it satisfies the available optimum. The process is repeated until all the secondary constraints are accounted for. Apply the proposed procedure to the following LP:

$$\begin{aligned}
 \mathbf{max} \quad & z = 5x_1 + 6x_2 + 3x_3 \\
 \mathbf{s.t.} \quad & 5x_1 + 5x_2 + 3x_3 \leq 50 \\
 & x_1 + x_2 - x_3 \leq 20 \\
 & 7x_1 + 6x_2 - 9x_3 \leq 30 \\
 & 5x_1 + 5x_2 + 5x_3 \leq 35 \\
 & 12x_1 + 6x_2 \leq 90 \\
 & x_2 - 9x_3 \leq 20 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

[ **Hint:** Start with the first, third, and fourth constraint. The associated solution is  $x_1 = 0, x_2 = 6.2, x_3 = 0.8$ . This solution automatically satisfies the second, fifth, and sixth constraint. Hence, these conditions are discarded as redundant and the optimal solution for the problem is  $x_1 = 0, x_2 = 6.2, x_3 = 0.8, z = 39.6$ . ]



## Answers

DON'T EVEN DARE PEEK AT THE SOLUTIONS TO AN EXERCISE UNTIL YOU'VE GENUINELY TRIED TO SOLVE THE EXERCISE !!

### Chapter 1 Exercise 1.1

1. Let  $x_1$  = number of hours of Process 1,  $x_2$  = number of hours of Process 2.
2. Then, the formulation of the problem is

$$\begin{array}{ll}\text{min} & w = 4x_1 + x_2 \\ \text{s.t.} & 3x_1 + x_2 \geq 10 \\ & x_1 + x_2 \geq 5 \\ & x_1 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

2. Let  $x_1$  = number of units of A,  $x_2$  = number of units of B. Then, the formulation of the problem is

$$\begin{array}{ll}\text{max} & z = 20x_1 + 50x_2 \\ \text{s.t.} & x_1 \geq 0.80(x_1 + x_2) \\ & x_1 \leq 100 \\ & 2x_1 + 4x_2 \leq 240 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

3. Let  $x_1$  = number of units of food F1 to be eaten,  $x_2$  = number of units of food F2 to be eaten. Then, the formulation of the problem is

$$\begin{array}{ll}\text{min} & w = 0.05x_1 + 0.03x_2 \\ \text{s.t.} & 2x_1 + x_2 \geq 400 \\ & x_1 + 2x_2 \geq 500 \\ & 4x_1 + 4x_2 \geq 1400 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

4. Let  $x_1$  = pounds of food A purchased,  $x_2$  = pounds of food B purchased. Then, the formulation of the problem is

$$\begin{array}{ll}\text{min} & w = 1.3x_1 + 0.8x_2 \\ \text{s.t.} & 5x_1 + 2x_2 \geq 60 \\ & 3x_1 + 2x_2 \geq 45 \\ & 4x_1 + x_2 \geq 30 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

5. Let  $x_1$ : number of computers of standard model,  $x_2$ : number of computers of portable model. Then the LP formulation is

$$\begin{array}{ll}\max & z = 320x_1 + 220x_2 \\ \text{s.t.} & 400x_1 + 250x_2 \leq 20000 \\ & 40x_1 + 30x_2 \leq 2160 \\ & x_1 \geq 0, x_2 \geq 0\end{array}$$

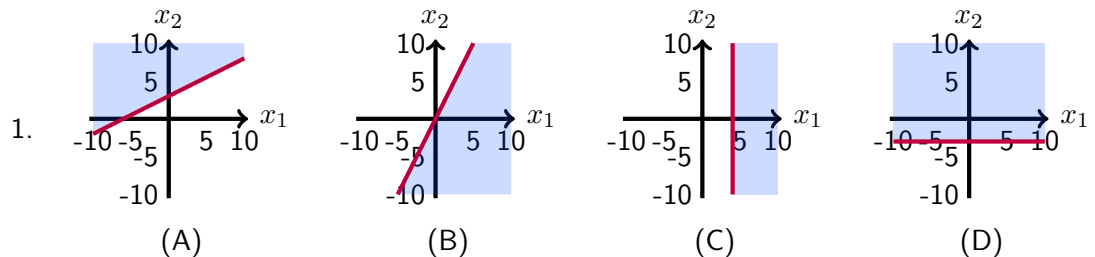
6. Let  $x_1$  = number of acres of crop A,  $x_2$  = number of acres of crop B, and  $x_3$  = number of acres of crop C. Then, the LP formulation is

$$\begin{array}{ll}\max & z = 100x_1 + 300x_2 + 200x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \leq 100 \\ & 40x_1 + 20x_2 + 30x_3 \leq 3200 \\ & x_1 + 2x_2 + x_3 \leq 160 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

7. Let  $x_1$  be the number of chocolate cakes baked, and  $x_2$  be the number of vanilla cakes baked. Then, the LP formulation is

$$\begin{array}{ll}\max & z = x_1 + 0.5x_2 \\ \text{s.t.} & 20x_1 + 40x_2 \leq 480 \\ & 4x_1 + x_2 \leq 30 \\ & x_1, x_2 \geq 0\end{array}$$

### Chapter 1 Exercise 1.2



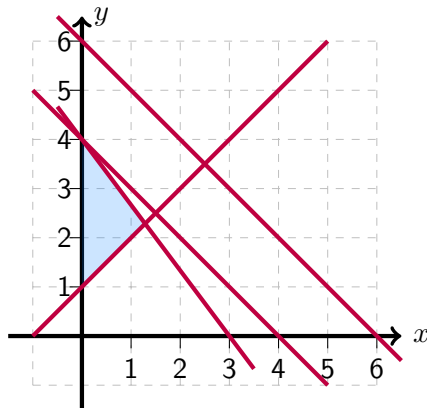
2. **Hyperplane:** (a), (e). **Half-Space:** (b). **Neither:** (c), (d), (f)

3. **Convex:** (A), (B), (F), (H)

4. (a)  $(1 + t, 3 + t, 2 - 3t)$  ;  $0 \leq t \leq 1$   
 (b)  $(1 + t, 3 + t, 2 - 3t) = (1.5, 3.5, 0.5)$  if  $t = 0.5 \in [0, 1]$

### Chapter 1 Exercise 1.3

1. (a) IV (b) II (c) I (d) III
2. (a) 16 (b) 55 (c) 84 (d) 90
3. (a) 32 (b) 55 (c) 36 (d) 32
4. (a)  $2a < b$  (c)  $b < \frac{a}{3}$  (e)  $b = \frac{a}{3}$   
 (b)  $\frac{a}{3} < b < 2a$  (d)  $b = 2a$
5. (a) south-east (down-right) (c) north-west (up-left)  
 (b) south-west (down-left)
6. (a) north-west (up-left) (b) south-east (down-right)
7. The constraints  $x + y \leq 6$  and  $x + y \leq 4$  are redundant.



8. The redundant constraint is  $y \leq 3$

$$-x + y \leq 1$$

$$x + y \leq 5$$

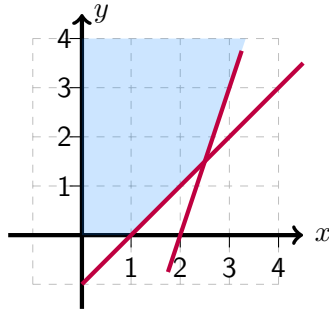
$$x - 2y \leq 2$$

$$y \leq 3$$

$$x \geq 1$$

$$x, y \geq 0$$

9.



10. *the solution is left to the student.*

11.  $x_1 = 2, x_2 = 3$  by solving the two constraint-equations.

12. (a)  $w = 12, x_1 = 3, x_2 = 0$  (d)  $z = 4, x_1 = 0, x_2 = 6$   
 (b) unbounded optimal solution (e)  $z = 16, x_1 = 6, x_2 = 2$   
 (c)  $w = 6, x_1 = 0, x_2 = 6$  (f)  $w = 14, x_1 = 3, x_2 = 2$

### Chapter 2 Exercise 2.1

1. **max**  $z = 2x_1 + 3x_2 + 5y_3 + 5y_4$   
**s.t.**  $-x_1 - x_2 + y_3 + y_4 + s_1 = 5$   
 $-6x_1 + 7x_2 - 9y_3 - 9y_4 + s_2 = 4$   
 $x_1 + x_2 + 4y_3 + 4y_4 = 10$   
 $x_1, x_2, y_3, y_4, s_1, s_2 \geq 0$
2.  $22x_1 - 4x_2 \geq -7 \Rightarrow 22x_1 - 4x_2 - e_1 = -7 \Rightarrow -22x_1 + 4x_2 + e_1 = 7$   
 $22x_1 - 4x_2 \geq -7 \Rightarrow -22x_1 + 4x_2 \leq 7 \Rightarrow -22x_1 + 4x_2 + s_1 = 7$
3. 

$x$	$y_1$	$y_2$
-6	0	6
10	10	0
0	0	6

### Chapter 2 Exercise 2.2

1. (a) **max**  $z = 2x_1 + 3x_2$   
**s.t.**  $x_1 + 3x_2 + s_1 = 12$   
 $3x_1 + 2x_2 + s_2 = 12$   
 $x_1, x_2, s_1, s_2 \geq 0$

(b)	NBVs	BVs	values	feasible?	$z$ -value
	$x_1, x_2$	$s_1, s_2$	$(12, 12)$	yes	0
	$x_2, s_2$	$s_1, x_1$	$(8, 4)$	yes	8
	$x_1, s_2$	$s_1, x_2$	$(-6, 6)$	no	
	$s_1, x_2$	$s_2, x_1$	$(-24, 12)$	no	
	$x_1, s_1$	$s_2, x_2$	$(4, 4)$	yes	12
	$s_1, s_2$	$x_1, x_2$	$(12/7, 24/7)$	yes	$96/7$

(c) from the table above, the optimum solution is  $z = 96/7$ ,  $x_1 = 12/7$ ,  $x_2 = 24/7$ .

(d) the solution is left to the student

(e) the solution is left to the student

2. (a) the solution is left to the student

(b)	BVs	values	feasible?	$w$ -value
	$x_1, x_3$	$(4, 0)$	yes	4
	$x_1, x_4$	$(4, 0)$	yes	4
	$x_2, x_3$	$(2, 0)$	yes	4
	$x_2, x_4$	$(2, 0)$	yes	4
	$x_3, x_4$	$(-4/7, 16/7)$	no	

3.	BVs	values	feasible?	$z$ -value
	$s_1, e_2$	$(3/2, -8)$	no	
	$s_1, x_1$	$(-1, 4)$	no	
	$s_1, x_2$	$(-13, 8)$	no	
	$e_2, x_1$	$(-5, 3)$	no	
	$e_2, x_2$	$(-13/2, 3/2)$	no	
	$x_1, x_2$	$(13/3, -2/3)$	no	

4. (a)	BVs	values	feasible?	$z$ -value
	$s_1, s_2$	$(2, 4)$	yes	0
	$s_1, x_1$	$(4, -2)$	yes	-2
	$s_1, x_2$	$(-2, 4)$	no	
	$s_2, x_1$	$(8, 2)$	yes	2
	$s_2, x_2$	$(2, 2)$	yes	6
	$x_1, x_2$	$(-2/3, 8/3)$	yes	$22/3$

(b) see the table above

(c) the solution is left to the student

### Chapter 2 Exercise 2.3

1. (a)  $z = 32/3$ ,  $x_1 = 10/3$ ,  $x_2 = 4/3$
- (b)  $z = 25$ ,  $x_1 = 15$ ,  $x_2 = 5$ ,  $x_3 = 0$
- (c)  $z = 25$ ,  $x_1 = 25$ ,  $x_2 = 0$

(d)  $z = 12, x_1 = 4, x_2 = 4, x_3 = 4$

2.	BVs	values	feasible?	$z$ -value
	$x_1$	30	yes	150
	$x_2$	10	yes	-60
	$x_3$	6	yes	18
	$x_4$	5	yes	-25
	$x_5$	10	yes	120
	$s_1$	30	yes	0

### Chapter 2 Exercise 2.4

1.  $w = -5, x_1 = 0, x_2 = 5$
2.  $w = -2, x_1 = 0, x_2 = 2$
3.  $w = -7.5, x_1 = 0, x_2 = 1.5$
4.  $w = -9, x_1 = 3, x_2 = 0$
5.  $w = -48, x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 8$

### Chapter 2 Exercise 2.5

1. (a)  $w = 1, x_1 = 0, x_2 = 0, x_3 = 1$   
 (b)  $w = 2, x_1 = 2, x_2 = 0, x_3 = 0$   
 (c)  $w = 4, x_1 = 2, x_2 = 0$   
 (d)  $z = 5, x_1 = 1, x_2 = 2$
2.  $w = 214/7, x_1 = 66/7, x_2 = 8/7, x_3 = 0, x_4 = 0$
3.  $z = 10, x_1 = 4, x_2 = 0, x_3 = 2$

### Chapter 2 Exercise 2.6

1. (a) iterations 2 and 3 are degenerate, and degeneracy is removed in iteration 4.  
 (b) *the solution is left to the student*  
 (c) 3 iterations
2.  $z = 4, x_1 = 4, x_2 = 0, x_3 = 0$
3.  $z = 10$  when:  $x_1 = 0, x_2 = 0, x_3 = 10/3$   
 $x_1 = 0, x_2 = 5, x_3 = 0$   
 $x_1 = 1, x_2 = 4, x_3 = 1/3$

4.  $x_3$  and  $s_1$  can yield alternative optima, but because all their constraint coefficients are non-positive, non can yield an alternative basic solution.

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS
$z$	0	0	0	0	1	20
$x_1$	1	0	-2	-1	0	15
$s_2$	0	1	-7	-2	1	10

5. the optimal solution is degenerate because  $s_3$  is basic and equal 0, also it has alternative nonbasic solution because  $s_2$  has zero coefficient in  $z$ -row and all its coefficients are  $\leq 0$ .

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	RHS
$z$	0	5	0	3	0	0	15
$x_3$	0	1	1	1	-1	0	3
$x_1$	1	2	0	1	0	0	5
$s_3$	0	-6	0	-2	-5	1	0

6. (a) solution space unbounded in the direction of  $x_3$   
 (b) objective value is unbounded because each unit increase in  $x_3$ , increases  $z$  by 1.
7. because  $a_2 = 0$  in the optimal tableau, the problem has feasible optimal solution:  $x_1 = 0$ ,  $x_2 = 4$ ,  $z = 8$ .

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$e_2$	$a_2$	RHS
$z$	$5M - 1$	0	$2M - 1$	$M$	$4M + 2$	0	8
$x_2$	2	1	1	0	1	0	4
$a_2$	-5	0	-2	-1	-4	1	0

8. (a) BVs :  $(x_8, x_3, x_1) = (12, 6, 0)$  ;  $z = 620$   
 NBVs :  $(x_2, x_4, x_5, x_6, x_7) = (0, 0, 0, 0, 0)$
- (b)  $x_2$  enters:  $x_2 = \min\left(\frac{12}{3}, \frac{6}{1}, -\right) = 4$ . so  $x_8$  leaves with  $\Delta z = 5 \times 4 = 20$ .  
 $x_5$  enters:  $x_5 = \min\left(-, \frac{6}{1}, \frac{0}{6}\right) = 0$ . so  $x_1$  leaves with  $\Delta z = 1 \times 0 = 0$ .  
 $x_6$  enters:  $x_2 = \min(-, -, -) = 4$ . so no leaving variable leaves and  $x_6$  can be increased to  $\infty$  with  $\Delta z = \infty$ .
- (c) the solution is left to the student
- (d)  $x_5$  and  $x_7$
9. (a)  $c < 0$ ,  $b \geq 0$   
 (b)  $c = 0$ ,  $b \geq 0$ ,  $a_2 > 0$  and/or  $a_3 > 0$ . If only  $a_3 > 0$  then  $b > 0$   
 (c)  $c > 0$ ,  $a_2 \leq 0$ ,  $a_3 \leq 0$

10. (a)  $b \geq 0$  is necessary.
- If  $c_1 = 0$  and  $c_2 \geq 0$  we can pivot in  $x_1$  to obtain an alternative optimum.
  - If  $c_1 \geq 0$ ,  $c_2 \geq 0$  and  $a_2 > 0$  we can pivot in  $x_5$  and obtain an alternative optimum.
  - If  $c_2 = 0$ ,  $a_1 > 0$  and  $c_1 \geq 0$  we can pivot in  $x_2$  and obtain an alternative optimum.
- (b)  $b < 0$
- (c)  $b = 0$
- (d)  $b \geq 0$  makes the solution feasible. If  $c_2 < 0$  and  $a_1 \leq 0$  we can make  $x_2$  as large as desired and obtained an unbounded solution.
- (e)  $b \geq 0$  makes the current basic solution feasible. For  $x_6$  to replace  $x_1$  we need  $c_1 < 0$  (this ensures that increasing  $x_1$  will increase  $z$ ) and we need Row 3 to win the ratio test for  $x_1$ . This requires  $3/a_3 \leq b/4$ .
11. *the solution is left to the student*

### Chapter 3 Exercise 3.1

1. Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
$z$	0	0	4	5	28
$x_1$	1	0	1	1	6
$x_2$	0	1	1	2	10

2. Basic	$x_1$	$x_2$	$s_1$	$s_2$	RHS
$z$	2	0	0	1	2
$x_2$	1	1	0	1	2
$s_1$	1	0	1	-1	2

3. *the solution is left to the student*

4. Basic	$x_1$	$x_2$	$e_1$	$e_2$	RHS
$z$	0	0	-5	$-15/2$	3800
$x_1$	1	0	$-3/20$	$1/40$	$18/5$
$x_2$	0	1	$1/40$	$-7/80$	$7/5$

### Chapter 3 Exercise 3.2

1.  $-\frac{100}{3} \leq \Delta \leq 300$  million
2.  $-24 \leq \Delta \leq 56$  million, and if  $\Delta = 40$  million then  $z_{new} = 520$  million
3. by adding the activity  $x_3 = \begin{bmatrix} 110 \\ 12 \\ 7 \end{bmatrix}$  the current solution is not optimal



### Chapter 3 Exercise 3.3

1. **min**  $w = y_1 + 3y_2 + 4y_3$   
**s.t.**  $-y_1 + y_2 + y_3 \geq 2$   
 $y_1 + y_2 - 2y_3 \geq 1$   
 $y_1, y_2, y_3 \geq 0$
2. **max**  $z = 4x_1 + x_2 + 3x_3$   
**s.t.**  $2x_1 + x_2 + x_3 \leq 1$   
 $x_1 + x_2 + x_3 \leq -1$   
 $x_1, x_2, x_3 \geq 0$
3. **min**  $w = 5y_1 + 7y_2 + 6y_3 + 4y_4$   
**s.t.**  $y_1 + 2y_2 + y_4 \geq 4$   
 $y_1 + y_2 + 2y_3 = -1$   
 $y_3 + y_4 = 2$   
 $y_1, y_2 \geq 0, y_3 \leq 0, x_4$  urs
4. **max**  $z = 6x_1 + 8x_2$   
**s.t.**  $x_1 + x_2 \leq 4$   
 $2x_1 - x_2 \leq 2$   
 $2x_2 = -1$   
 $x_1 \leq, x_2$  urs

### Chapter 3 Exercise 3.4

1.  $w = 250/3$ .
2. (a) *the solution is left to the student*  
 (b)  $\bar{\mathbf{C}}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} = [-2 \quad 7/2]$   
 (c) •  $\mathbf{y} = \mathbf{C}_{BV}\mathbf{B}^{-1} = [4 \quad 0] = [y_1 \quad y_2]$   
 • Using the general formula:  $y_1 = 0 + 4 = 4$   
 $y_2 = 3 + (-3) = 0$
3.  $\bar{z} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\bar{\mathbf{y}}} \mathbf{b} = [0.4 \quad 1.4] \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 0.9$
4. since  $\bar{\mathbf{y}} = [0 \quad 1]$  then  $\bar{\mathbf{w}} = 10 \neq 20/3$
5.  $x_1 = 0, x_2 = 20, x_3 = 0, z = 1200$
6. *the solution is left to the student*

### Chapter 3 Exercise 3.5

1.  $y_1 = 4, y_2 = 1$
2.  $z_{\text{new}} = z_{\text{old}} + 10 \times 4 = 340$

### Chapter 3 Exercise 3.6

1. (a) dual constraint for  $x_1$  is  $y_1 + 2y_2 \geq c_1$ . since  $y_1 = 4$  and  $y_2 = 1$  the current basis is still optimal if  $c_1 \leq 6$   
 (b) dual constraint for Type 1 Candy Bar is now  $\frac{y_1}{2} + \frac{3y_2}{4} \geq 3$ . since  $y_1 = 4$  and  $y_2 = 1$  does not satisfy this constraint, the current basis is no longer optimal

- (c) the dual constraint for a Type 4 Candy Bar is  $2y_1 + y_2 \geq 10$ . since  $y_1 = 4$  and  $y_2 = 1$  does not satisfy this constraint, the current basis is no longer optimal. the new optimal solution would make Type 4 Candy Bars.
2. (a) *the solution is left to the student*  
 (b)  $c_2 \leq 1/6$   
 (c)  $c_1 \leq 5/6$

### Chapter 3 Exercise 3.7

$$y_1 = 1, y_2 = 1, \bar{w} = 13$$

### Chapter 3 Exercise 3.8

1. (a)  $x_1 = 4/3, x_2 = 10/3, x_3 = 0, z = 2/3$   
 (b) *the solution is left to the student*
2.  $x_1 = 0, x_2 = 14, x_3 = 9, z = -9$
3.  $x_1 = 0, x_2 = 10, x_3 = 0, x_4 = 0, w = 70$
4. *the solution is left to the student*