Lecture Notes for Calculus 101

Chapter 0 : Before Calculus

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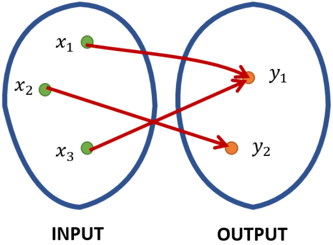
Idea and Definition

Definition 1

A function f is a rule that associates with each input, a unique (exactly ONE) output.



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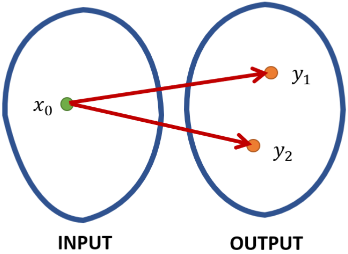

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Idea and Definition

Example 1

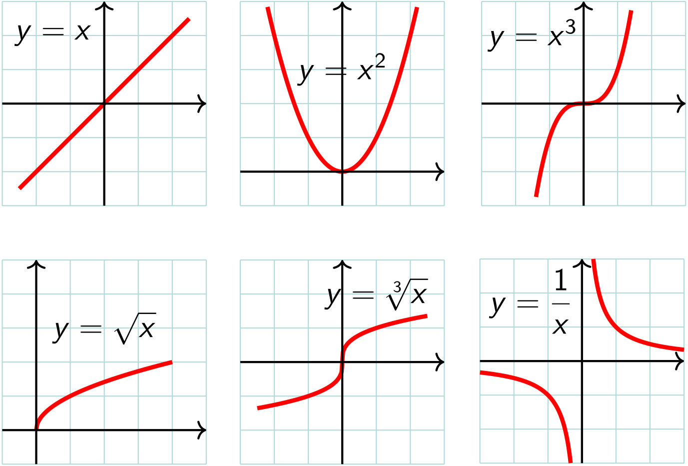
Let $f(x) = 3x^2 - 4x + 2$. Then

$$\begin{aligned} f(-1) &= 3(-1)^2 - 4(-1) + 2 \\ &= 3 + 4 + 2 = 9 \end{aligned}$$

NOTE: **There are 4-ways to represent functions:**

- Table of values
- Words
- Graph
- Formula

Some Graphs

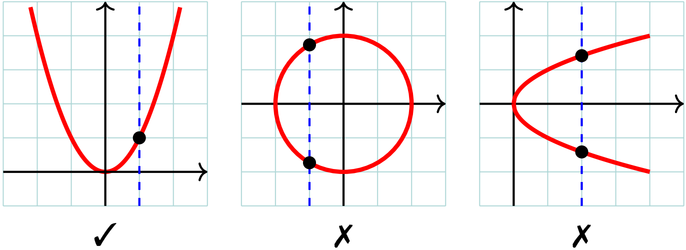


Vertical Line Test

A curve in the $xy-$plane is the graph of some function f if and only if no vertical line intersects the curve more than once

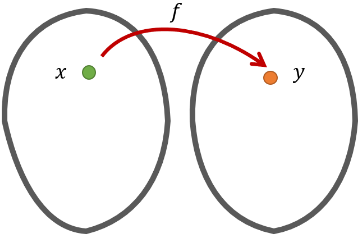
Example 2

Which of the following graphs is a function?



Domain (and Range) of Functions

- The set of all allowable inputs is the **Domain** of f.
- The set of all resulting outputs is the **Range** of f.



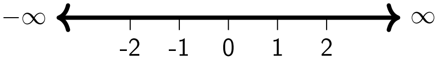
NOTE: If no domain is mentioned, we always assume the largest possible domain, called the **Natural Domain**.

Examples on Function Domain

Example 3

Find the natural domain of the function $f(x) = x^3 - 2x^2 + 1$.

$$\begin{aligned} \text{dom}(f) &= \text{all real numbers} \\ &= (-\infty, \infty) \\ &= \mathbb{R} \end{aligned}$$



NOTE: Any polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has domain $\mathbb{R}$.

Examples on Function Domain

Example 4

Find the natural domain of the function $f(x) = \dfrac{1}{(x+1)(x-3)}$.

$$\begin{aligned} \operatorname{dom}(f) &= \text{all real numbers except } -1 \text{ and } 3 \\ &= \mathbb{R} - \{-1, 3\} \\ &= (-\infty, -1) \cup (-1, 3) \cup (3, \infty) \end{aligned}$$

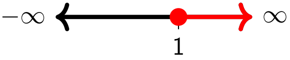
NOTE: For rational functions, avoid division by 0.

Examples on Function Domain

Example 5

Find the natural domain of the function $f(x) = \sqrt{x-1}$.

$$\begin{aligned} \operatorname{dom}(f) &= \text{all real numbers such that } x-1 \geq 0 \\ &= \text{all } x \in \mathbb{R} \text{ such that } x \geq 1 \\ &= [1, \infty) \end{aligned}$$



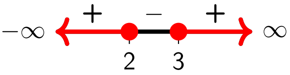
NOTE: Avoid even roots of negative numbers.

Examples on Function Domain

Example 6

Find the natural domain of the function $f(x) = \sqrt{x^2 - 5x + 6}$.

$$\text{dom}(f) = \text{all real numbers such that} \quad x^2 - 5x + 6 \geq 0$$
$$(x-2)(x-3) \geq 0$$
$$= (-\infty, 2] \cup [3, \infty)$$



Examples on Function Domain

Example 7

Find the natural domain of the function $f(x) = \sqrt[3]{4 - 3x^2}$.

$$\begin{aligned} \text{dom}(f) &= \text{all real numbers} \\ &= \mathbb{R} \end{aligned}$$

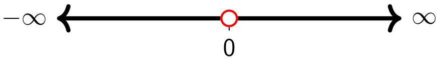
NOTE: Odd root accepts $+$, $-$, 0 but depends on the function inside it.

Examples on Function Domain

Example 8

Find the natural domain of the function $f(x) = \sqrt[3]{\frac{1}{x}}$.

$$\begin{aligned} \text{dom}(f) &= \text{all real numbers except } 0 \\ &= (-\infty, 0) \cup (0, \infty) \end{aligned}$$

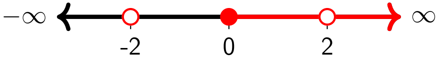


Examples on Function Domain

Example 9

Find the natural domain of the function $f(x) = \dfrac{\sqrt{x}}{x^2 - 4}$.

$$\begin{aligned} \text{dom}(f) &= \text{all real numbers such that } x \geq 0 \text{ except } \pm 2 \\ &= [0, 2) \cup (2, \infty) \end{aligned}$$



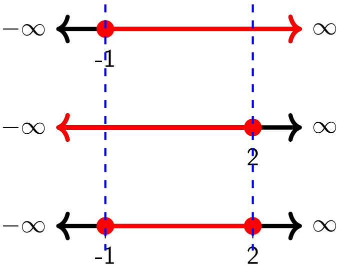
Examples on Function Domain

Example 10

Find the natural domain of the function $f(x) = \sqrt{x+1} - \sqrt{2-x}$.

$$\begin{aligned} \text{dom}\left(\sqrt{x+1}\right) &= x \in \mathbb{R}; x+1 \geq 0 \\ &\qquad\qquad x \geq -1 \\ &= [-1, \infty) \\ \text{dom}\left(\sqrt{2-x}\right) &= x \in \mathbb{R}; 2-x \geq 0 \\ &\qquad\quad -x \geq -2 \\ &\qquad\qquad x \leq 2 \\ &= (-\infty, 2] \end{aligned}$$

$$\therefore \text{dom}(f) = [-1, \infty) \cap (-\infty, 2] = [-1, 2]$$



Examples on Function Domain

Exercise 1

(1) *Compare the domain of* $g(x) = x$ *and* $f(x) = \dfrac{x^2 + x}{1 + x}$.

(2) *Find the domain of each of the following.*

a) $f(x) = \sqrt{x^2 + 2x + 4}$

b) $g(x) = \dfrac{x}{x^2 - x - 2}$

c) $f(x) = \dfrac{\sqrt{9 - x^2}}{1 - x^2}$

d) $g(x) = \dfrac{x - 3}{1 - \sqrt{x}}$

e) $f(x) = \dfrac{x - 3}{1 + \sqrt{x}}$

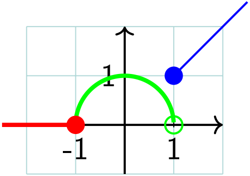
Piecewise Functions

Functions can be defined in pieces (cases).

Example 11

Let $f(x) = \begin{cases} 0 & : \quad x \leq -1 \\ \sqrt{1-x^2} & : \quad -1 < x < 1 \\ x & : \quad x \geq 1 \end{cases}$

$$f(0) = \sqrt{1-0^2} = 1$$
$$f(-3) = 0$$
$$f(5) = 5$$
$$f(1) = 1$$



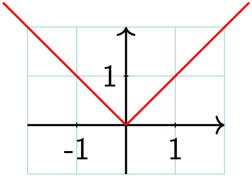
NOTE: $x = -1$ and $x = 1$ are called piecewise points.

Absolute Value

The absolute value of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & : \quad x \geq 0 \\ -x & : \quad x < 0 \end{cases}$$

For example, $|4| = 4$, $|0| = 0$, $\left|-\frac{1}{2}\right| = \frac{1}{2}$.



NOTE:

(1) $|x| = a \Leftrightarrow x = \pm a$. **Ex.** $|x| = 4 \Leftrightarrow x = \pm 4$

(2) $|x| \leq a \Leftrightarrow -a \leq x \leq a$. **Ex.** $|x| \leq 4 \Leftrightarrow -4 \leq x \leq 4$

(3) $|x| \geq a \Leftrightarrow x \geq a$ or $x \leq -a$. **Ex.** $|x| \geq 4 \Leftrightarrow x \leq -4$ or $x \geq 4$

(4) $\sqrt{x^2} = |x|$

(5) $\text{dom}|g(x)| = \text{dom}(g(x))$

Absolute Value

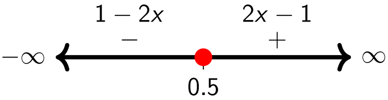
Example 12

Write the function $g(x) = |2x-1|$ as piecewise function.

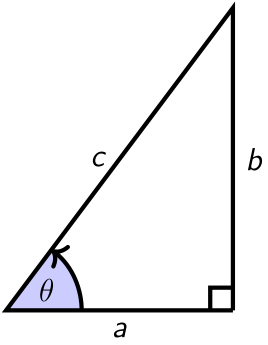
$$g(x) = \begin{cases} 2x-1 & : \quad 2x-1 \geq 0 \\ -(2x-1) & : \quad 2x-1 < 0 \end{cases}$$

$$= \begin{cases} 2x-1 & : \quad x \geq 1/2 \\ -2x+1 & : \quad x < 1/2 \end{cases}$$

$$g(x) = 0 \Rightarrow |2x-1| = 0$$
$$\Rightarrow 2x - 1 = 0$$
$$\Rightarrow x = 1/2$$



Trigonometric Ratios (Right Triangle)



a : **adjacent**

b : **opposite**

c : **hypotenuse**

$$\sin\theta = \frac{b}{c} \quad , \quad \cos\theta = \frac{a}{c} \quad , \quad \tan\theta = \frac{b}{a}$$

$$\sec\theta = \frac{c}{a} \quad , \quad \csc\theta = \frac{c}{b} \quad , \quad \cot\theta = \frac{a}{b}$$

NOTE:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Angle Measures and Standard Position

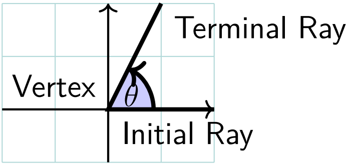
We measure angles either by degrees or by radians, where

$$\pi \text{ rad} = 180^\circ$$

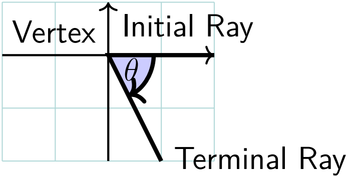
Example 13

$$90^\circ = 90 \times \frac{\pi}{180} = \frac{\pi}{2}$$

$$\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi} = 120^\circ$$



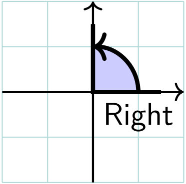
Counterclockwise
θ is positive



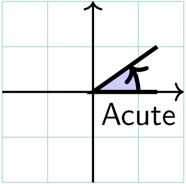
Clockwise
θ is negative

Angle Measures and Standard Position

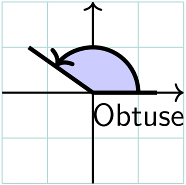
Example 14



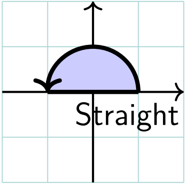
$90^\circ = \pi/2$



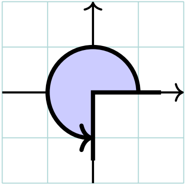
$30^\circ = \pi/6$



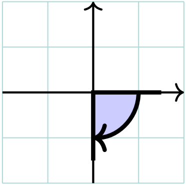
$150^\circ = 5\pi/6$



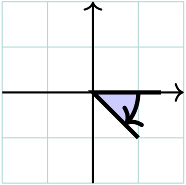
$180^\circ = \pi$



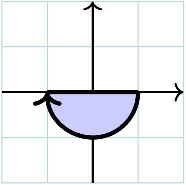
$270^\circ = 3\pi/2$



$-90^\circ = -\pi/2$



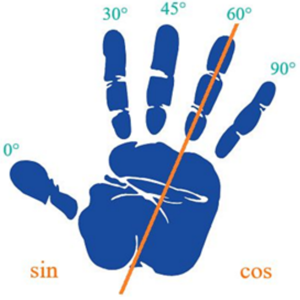
$-45^\circ = -\pi/4$



$-540^\circ = -3\pi$

Some Trigonometric Values

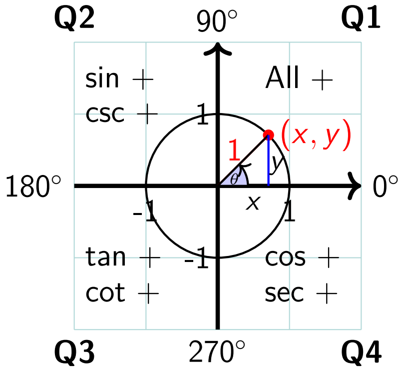
θ deg	0°	30°	45°	60°	90°
θ rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	✗
$\csc\theta$	✗	2	$\sqrt{2}$	$2/\sqrt{3}$	1
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	✗
$\cot\theta$	✗	$\sqrt{3}$	1	$1/\sqrt{3}$	0



	0°	30°	45°	60°	90°
sin	0	1	2	3	4
cos	4	3	2	1	0

(Each value under $\sqrt{\ }$, divided by 2)

Trigonometric Values in Coordinate Plane



$(x, y) = (\cos\theta, \sin\theta)$

4 Quadrants

θ	Point	sin	cos
0°	$(1, 0)$	0	1
90°	$(0, 1)$	1	0
180°	$(-1, 0)$	0	-1
270°	$(0, -1)$	-1	0

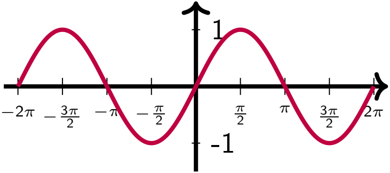
Trigonometric Functions

① $f(x) = \sin x$

Domain : $\mathbb{R}$

Range : $\left[-1, 1\right]$

Period : 2π



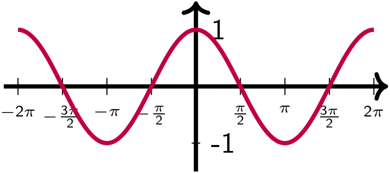
Trigonometric Functions

(2) $f(x) = \cos x$

Domain : $\mathbb{R}$

Range : $\big[-1, 1\big]$

Period : 2π



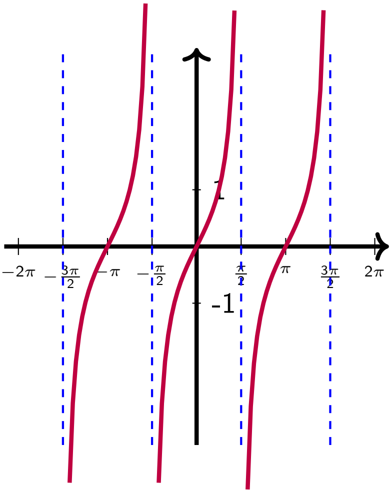
Trigonometric Functions

③ $f(x) = \tan x$

Domain : $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

Range : $\mathbb{R}$

Period : π



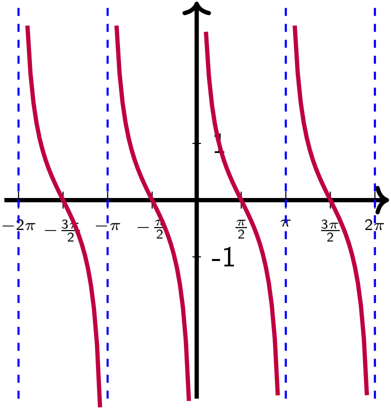
Trigonometric Functions

(4) $f(x) = \cot x$

Domain : $x \neq 0, \pm\pi, \pm 2\pi, \cdots$

Range : $\mathbb{R}$

Period : π



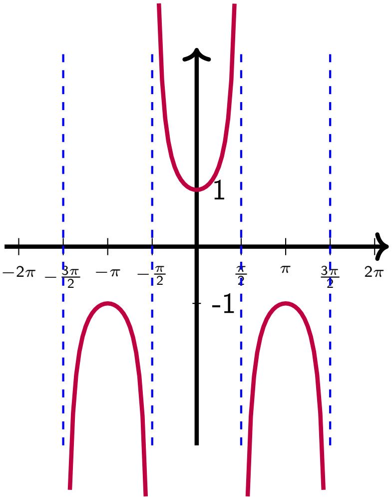
Trigonometric Functions

(5) $f(x) = \sec x$

Domain : $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

Range : $(-\infty, -1] \cup [1, \infty)$

Period : 2π



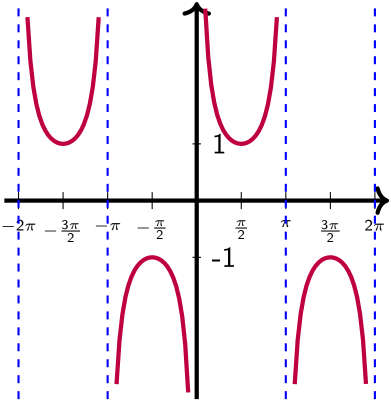
Trigonometric Functions

6 $f(x) = \csc x$

Domain : $x \neq 0, \pm\pi, \pm 2\pi, \cdots$

Range : $(-\infty, -1] \cup [1, \infty)$

Period : 2π



Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\begin{aligned}\sin(-\theta) &= -\sin\theta\\ \cos(-\theta) &= \cos\theta\\ \tan(-\theta) &= -\tan\theta\end{aligned}$$

Ex: $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

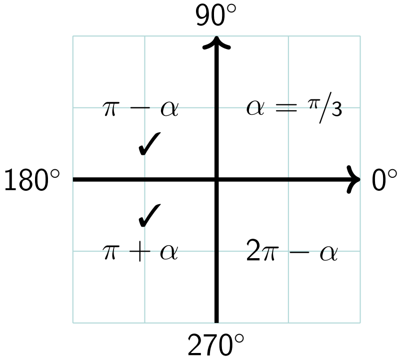
Reference Angle

Example 15

Find all angles θ such that $\cos\theta = -1/2$.

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \pm 2n\pi$$
$$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \pm 2n\pi$$

where $n = 0, 1, 2, \cdots$



Reference Angle

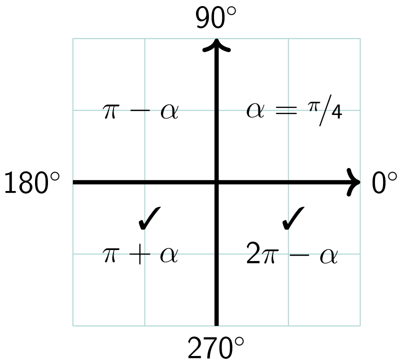
Example 16

Find all angles θ such that $\sin\theta = -1/\sqrt{2}$.

$$\theta_1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \pm 2n\pi$$

$$\theta_2 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \pm 2n\pi$$

where $n = 0, 1, 2, \cdots$



Reference Angle

Example 17

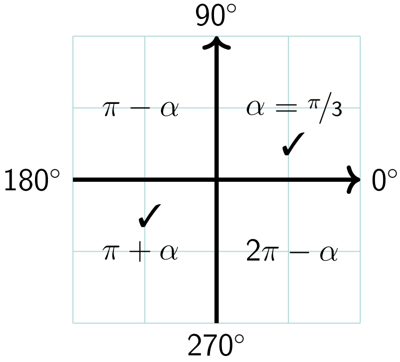
Find all angles θ such that $\tan\theta = \sqrt{3}$.

$$\theta_1 = \frac{\pi}{3} \pm 2n\pi$$

$$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \pm 2n\pi$$

$$\therefore \theta = \frac{\pi}{3} \pm n\pi$$

where $n = 0, 1, 2, \cdots$



Reference Angle

Example 18

Find all angles θ such that $\sin\theta = 0$.

$$\theta_1 = 0 \pm 2n\pi$$
$$\theta_2 = \pi \pm 2n\pi$$
$$\therefore \theta = 0 \pm n\pi = \pm n\pi$$

where $n = 0, 1, 2, \cdots$

Reference Angle

Example 19

Find all angles θ such that $\cos\theta = 0$.

$$\theta_1 = \frac{\pi}{2} \pm 2n\pi$$
$$\theta_2 = \frac{3\pi}{2} \pm 2n\pi$$
$$\therefore \theta = \frac{\pi}{2} \pm n\pi$$

where $n = 0, 1, 2, \cdots$

Reference Angle

Example 20

Find all angles θ such that $\cos\theta = -1$.

$$\theta = \pi \pm 2n\pi = \left(1 \pm 2n\right)\pi \text{ where } n = 0, 1, 2, \cdots$$

Example 21

Find all angles θ such that $\sin\theta = 1$.

$$\theta = \frac{\pi}{2} \pm 2n\pi \text{ where } n = 0, 1, 2, \cdots$$

Reference Angle

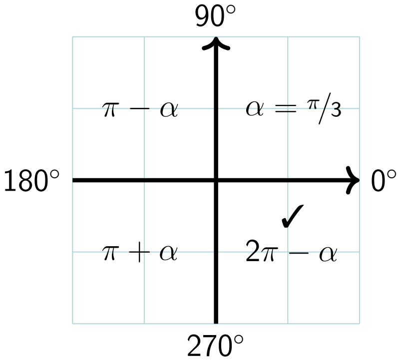
Example 22

Find the value of $\sin\left(\frac{5\pi}{3}\right)$.

$$2\pi - \alpha = \frac{5\pi}{3}$$

$$\alpha = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

$$\therefore \sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$



$^{5\pi}/_{3} = 300°$

Reference Angle

Example 23

Find the value of

$\tan\left(\frac{-3\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right)$

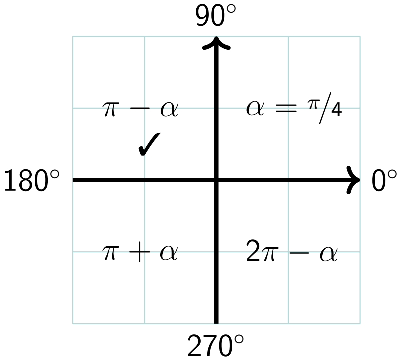
$$\pi - \alpha = \frac{3\pi}{4}$$

$$\alpha = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

$$\therefore \tan\left(\frac{-3\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right)$$

$$= -(-)\tan\left(\frac{\pi}{4}\right)$$

$$= 1$$



$^{3\pi}/_4 = 135^\circ$

Reference Angle

Example 24

Find the domain of $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$$\begin{aligned}\text{dom}(\tan x) &= \mathbb{R} - \Big\{ \cos x = 0 \Big\} \\ &= \mathbb{R} - \Big\{ x = {}^{\pi}/_{2} \pm n\pi \Big\} = \text{dom}(\sec x)\end{aligned}$$

Example 25

Find the domain of $f(x) = \csc x = \dfrac{1}{\sin x}$

$$\begin{aligned}\text{dom}(\csc x) &= \mathbb{R} - \Big\{ \sin x = 0 \Big\} \\ &= \mathbb{R} - \Big\{ x = \pm n\pi \Big\} = \text{dom}(\cot x)\end{aligned}$$

Operations on Functions

Let f and g be functions, then

$$\left(f \overset{\times}{\underset{-}{+}} g\right)(x) = f(x) \overset{\times}{\underset{-}{+}} g(x)\ ;\ x \in \text{dom}(f) \cap \text{dom}(g)$$

$$(f \div g)(x) = f(x) \div g(x)\ ;\ x \in \text{dom}(f) \cap \text{dom}(g) - \left\{g(x) = 0\right\}$$

Example 26

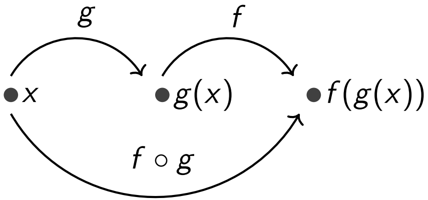
Let $f(x) = \sqrt{x} \to \text{dom}(f) = [0, \infty)$. Find $(f \cdot g)(x)$ and its domain.

$g(x) = \sqrt{x} \to \text{dom}(g) = [0, \infty)$

$$(f \cdot g)(x) = \sqrt{x} \cdot \sqrt{x} = \left(\sqrt{x}\right)^2 = x$$

$$\text{dom}(f \cdot g) = [0, \infty) \cap [0, \infty) = [0, \infty)$$

Composition of Functions



f "**circle**" g of x
f "**after**" g of x

Definition 2

$(f \circ g)(x) = f(g(x))$

$\text{dom}(f \circ g) =$ set of all $x \in \text{dom}(g)$ such that $g(x) \in \text{dom}(f)$

NOTE: In general, $f \circ g \neq g \circ f$.

Composition of Functions

Example 27

Given that $f(1)=1$, $f(-1)=0$, $g(-1)=1$ and $g(0)=0$. Find $(f\circ g)(-1)$.

$$(f\circ g)(-1)=f(g(-1))=f(1)=1$$

Example 28

Let $f(x)=x^2+3$ and $g(x)=\sqrt{x}$. then

(1) $(f\circ g)(x)=f(g(x))=f(\sqrt{x})=\left(\sqrt{x}\right)^2+3=x+3$

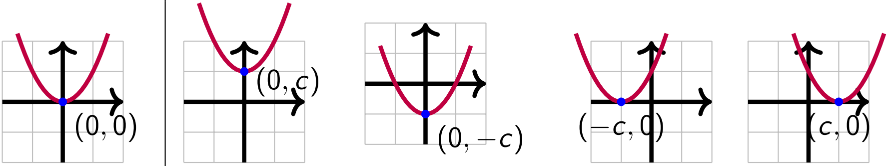
$\text{dom}(f\circ g)=[0,\infty)\cap\mathbb{R}=[0,\infty)$

(2) $(g\circ f)(x)=g(f(x))=g(x^2+3)=\sqrt{x^2+3}$

$\text{dom}(g\circ f)=\mathbb{R}\cap\mathbb{R}=\mathbb{R}$

How Operations Affect Functions Graphs (Translation)

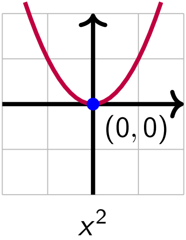
Example	$f(x)+c$	$f(x)-c$	$f(x+c)$	$f(x-c)$
$f(x)=x^2$	**up**	**down**	**left**	**right**
	x^2+c	x^2-c	$(x+c)^2$	$(x-c)^2$

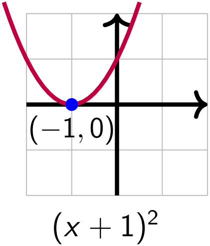


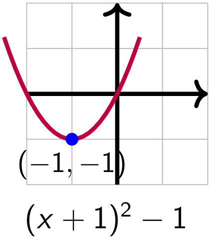
How Operations Affect Functions Graphs (Translation)

Example 29

Draw the graph of $f(x) = (x+1)^2 - 1$.

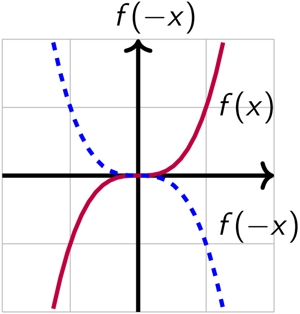




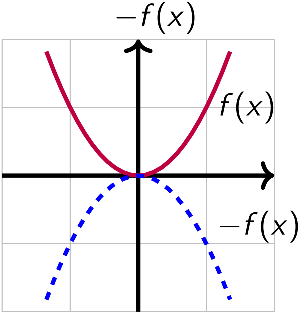


How Operations Affect Functions Graphs (Reflection)

About $y-$axis



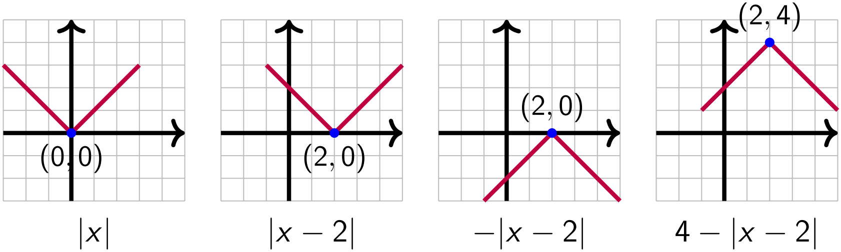
About $x-$axis



How Operations Affect Functions Graphs (Reflection)

Example 30

Draw the graph of $f(x) = 4 - |x - 2|$



How Operations Affect Functions Graphs (Reflection)

NOTE: Follow this order of transformations:

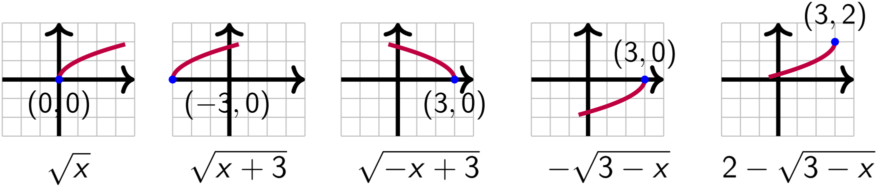
(1) Horizontal shifts: $f(x \pm c)$

(2) Reflection: $f(-x)$ and/or $-f(x)$

(3) Vertical shifts: $f(x) \pm c$

Example 31

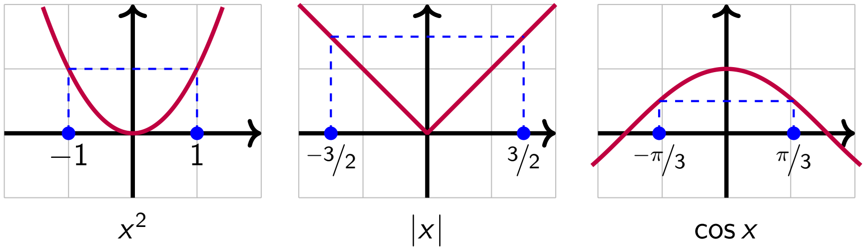
Draw the graph of $f(x) = 2 - \sqrt{3 - x}$



Functions with Symmetric Graphs

Definition 3

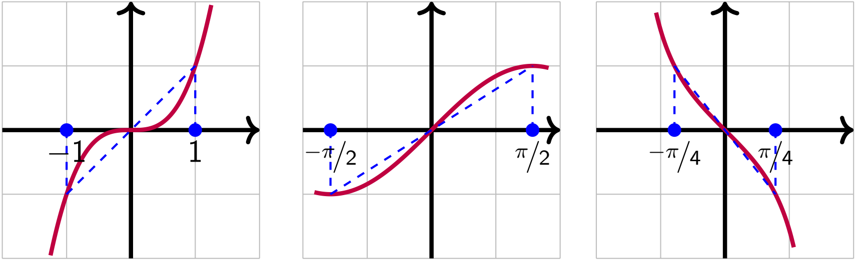
$f(x)$ is an **even function** if $f(-x) = f(x)$. In this case, f is symmetric about the $y-$axis.



Functions with Symmetric Graphs

Definition 4

$f(x)$ is an **odd function** if $f(-x) = -f(x)$. In this case, f is symmetric about the origin.



x^3 $\qquad$ $\sin x$ $\qquad$ $-\tan x = \tan(-x)$

Functions with Symmetric Graphs

NOTE:

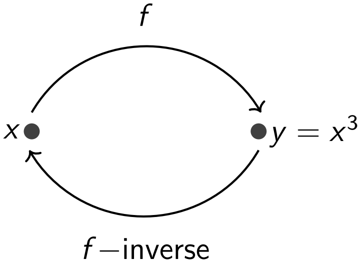
(1) Some functions are Odd, some are Even, and some neither Odd nor Even like $f(x) = x + 2\cos x$.

(2)
- Odd $\pm$ Odd $=$ Odd
- Even $\pm$ Even $=$ Even
- Odd $\times$ Odd $=$ Even , Odd $\div$ Odd $=$ Even
- Even $\times$ Even $=$ Even , Even $\div$ Even $=$ Even
- Odd $\times$ Even $=$ Odd , Odd $\div$ Even $=$ Odd

Idea of Inverse Functions

When two functions "UNDO" each other.

The inverse function is denoted by f^{-1} and read $f-$inverse.



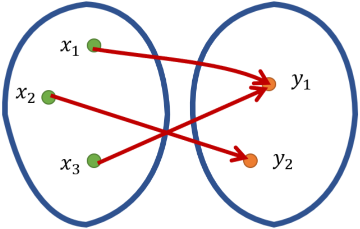
NOTE:

* $f^{-1} \neq \dfrac{1}{f}$
* If $y = x^3$, then $x = \sqrt[3]{y}$
* The inputs of f, are the outputs of f^{-1} and vice-versa.

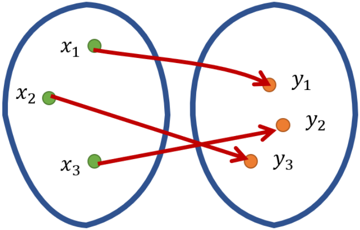
$$\text{domain } \left(f^{-1}\right) = \text{range } (f)$$

$$\text{range } \left(f^{-1}\right) = \text{domain } (f)$$

When Do the Inverse Exist?



Function but NOT $1-1$



Function and $1-1$

A function f has inverse $\Leftrightarrow$ it is one-to-one $(1-1)$

$\Leftrightarrow x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

$\Leftrightarrow$ The graph intersects any horizontal line at most once (*Horizontal Line Test*)

When Do the Inverse Exist?

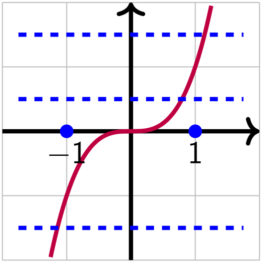
Example 32

Let $f(x) = x^3$.

* The domain of f is $\mathbb{R}$.
* If we take $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$, then

$$x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_1)$$

* So, x^3 is $1-1$ on $\mathbb{R}$.

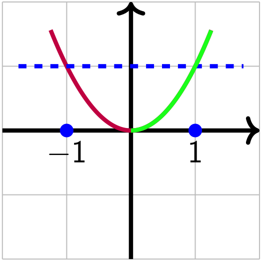


When Do the Inverse Exist?

Example 33

Let $f(x) = x^2$.

* The domain of f is $\mathbb{R}$.
* Note that $-1 \neq 1$, but $(-1)^2 = (1)^2$.
* So, x^2 is **NOT** $1-1$ on $\mathbb{R}$.
* But, x^2 is $1-1$ on $[0, \infty)$.



Definition of Inverse Function

Definition 5

The functions f, f^{-1} are inverses provided both

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x \ ; \ x \in \operatorname{dom}\left(f^{-1}\right)$$
$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x \ ; \ x \in \operatorname{dom}\left(f\right)$$

Example 34

Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

$$f^{-1}(x) = 1 \Rightarrow f\left(f^{-1}(x)\right) = f\left(1\right)$$
$$\Rightarrow x = f(1) = 10$$

Finding the Inverse Function

To find the inverse function of $f(x)$:

(1) Write $y = f(x)$.

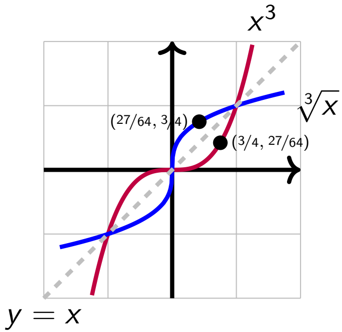
(2) Solve the equation for x as a function of y.

(3) Replace x by $f^{-1}(x)$, and y by x.

Example 35

Find $f^{-1}(x)$ given that $f(x) = x^3$.

$$y = x^3$$
$$\sqrt[3]{y} = \sqrt[3]{x^3}$$
$$x = \sqrt[3]{y}$$
$$f^{-1}(x) = \sqrt[3]{x}$$

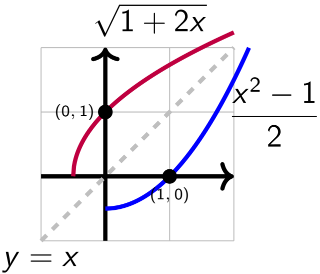


Finding the Inverse Function

Example 36

Find $f^{-1}(x)$ given that $f(x) = \sqrt{1+2x}$.

$$y = \sqrt{1+2x} \Rightarrow y^2 = \left(\sqrt{1+2x}\right)^2$$
$$\Rightarrow y^2 = 1 + 2x$$
$$\Rightarrow x = \frac{y^2 - 1}{2}$$
$$\Rightarrow f^{-1}(x) = \frac{1}{2}\left(x^2 - 1\right)$$



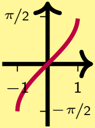
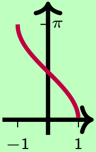
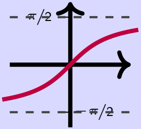
Finding the Inverse Function

Example 37

Find $f^{-1}(x)$ given that $f(x) = \dfrac{x+1}{x-1}$.

$$\begin{aligned} y = \frac{x+1}{x-1} &\Rightarrow (x-1)y = x+1 \\ &\Rightarrow xy - y = x+1 \\ &\Rightarrow xy - x = y+1 \\ &\Rightarrow x(y-1) = y+1 \\ &\Rightarrow x = \frac{y+1}{y-1} \Rightarrow f^{-1}(x) = \frac{x+1}{x-1} \end{aligned}$$

Inverse Trigonometric Functions

f^{-1}	**Domain**	**Range**	**Graph**	**Note**
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$		$\arcsin x$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$		$\arccos x$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		$\arctan x$

Inverse Trigonometric Functions

Example 38

Find the domain of $f(x) = \sin^{-1}(x-1)$.

$$-1 \leq x-1 \leq 1 \Rightarrow \underset{+1}{-1} \leq x\underset{+1}{-1} \leq \underset{+1}{1}$$
$$\Rightarrow 0 \leq x \leq 2 \Rightarrow \text{dom}(f) = [0,2]$$

Example 39

Find the exact value.

(1) $\sin^{-1}(1/2) = \pi/6 \in [-\pi/2, \pi/2]$

(2) $\cos^{-1}(1/2) = \pi/3 \in [0, \pi]$

(3) $\tan^{-1}(1) = \pi/4 \in (-\pi/2, \pi/2)$

(4) $\cos^{-1}(2)$ undefined

Inverse Trigonometric Functions

NOTE:

- * $\sin^{-1}(-x) = -\sin^{-1} x$
- * $\tan^{-1}(-x) = -\tan^{-1} x$
- * $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- * $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$
- * $\tan^{-1}(x) \neq \dfrac{\sin^{-1}(x)}{\cos^{-1}(x)}$

Example 40

Find the exact value.

(1) $\sin^{-1}\left(^{-1}/_{2}\right) = -\sin^{-1}\left(^{1}/_{2}\right) = -^{\pi}/_{6} \in \left[^{-\pi}/_{2}, ^{\pi}/_{2}\right]$

(2) $\tan^{-1}(-1) = -\tan^{-1}(1) = -^{\pi}/_{4} \in \left(^{-\pi}/_{2}, ^{\pi}/_{2}\right)$

(3) $\cos^{-1}\left(^{-1}/_{2}\right) = \pi - \cos^{-1}\left(^{1}/_{2}\right) = \pi - ^{\pi}/_{3} = ^{2\pi}/_{3} \in [0, \pi]$

Inverse Trigonometric Functions

Example 41

Find the exact value.

(1) $\sin\left(\sin^{-1}(1/4)\right) = 1/4$
Note that $\text{dom}\left(\sin^{-1}\right) = [-1, 1]$ and $1/4 \in [-1, 1]$
Also, sin and $\sin^{-1}$ are inverses and cancel each other

(2) $\tan(\tan^{-1}(-17/9)) = -17/9$
Note that $\text{dom}(\tan^{-1}) = \mathbb{R}$ and $-17/9 \in \mathbb{R}$
Also, tan and $\tan^{-1}$ are inverses and cancel each other

(3) $\cos(\cos^{-1}(-2/3)) = -2/3$
Note that $\text{dom}(\cos^{-1}) = [-1, 1]$ and $-2/3 \in [-1, 1]$
Also, cos and $\cos^{-1}$ are inverses and cancel each other

Inverse Trigonometric Functions

NOTE: $$\sin^{-1}(\sin x) = \begin{cases} \pi - x & : \ \pi/2 \le x \le 3\pi/2 \\ x - 2n\pi & : \ x \ge 3\pi/2 \end{cases}$$

$$\cos^{-1}(\cos x) = 2n\pi - x \ ; \quad \text{if } x \ge \pi$$

Example 42

Find the exact value.

(1) $\sin^{-1}\left(\sin\left(5\pi/3\right)\right) = \sin^{-1}\left(\sin\left(5\pi/3 - 2\pi\right)\right) = \sin^{-1}\left(\sin\left(-\pi/3\right)\right)$
$= -\pi/3$

(2) $\sin^{-1}\left(\sin\left(4\pi/3\right)\right) = \sin^{-1}\left(\sin\left(\pi - 4\pi/3\right)\right) = \sin^{-1}\left(\sin\left(-\pi/3\right)\right)$
$= -\pi/3$

(3) $\cos^{-1}\left(\cos\left(17\pi/4\right)\right) = \cos^{-1}\left(\cos\left(4\pi - 17\pi/4\right)\right)$
$= \cos^{-1}\left(\cos\left(-\pi/4\right)\right) = \cos^{-1}\left(\cos\left(\pi/4\right)\right) = \pi/4$

Inverse Trigonometric Functions

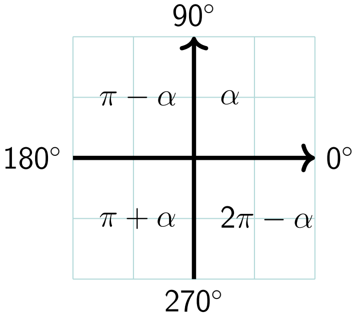
Example 43

Find the exact value of $\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$.

$$\pi - \alpha = \frac{5\pi}{6}$$

$$\alpha = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

$$\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right) = -\tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right) = \frac{-\pi}{6}$$

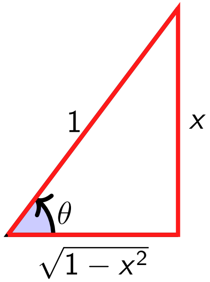


Inverse Trigonometric Functions

Example 44

Simplify the expression $\tan\left(\sin^{-1} x\right)$.

Let
$$\theta = \sin^{-1} x$$
$$\sin\theta = \sin\left(\sin^{-1} x\right)$$
$$\sin\theta = x \ ; \ x \in [-1, 1]$$
$$\therefore \tan\left(\sin^{-1} x\right) = \tan\theta = \frac{x}{\sqrt{1 - x^2}}$$
where $x \in (-1, 1)$

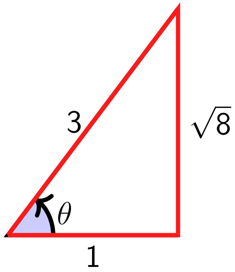


Inverse Trigonometric Functions

Example 45

Find the exact value of $\sin\left(2\sec^{-1}3\right)$.
Let

$$\begin{aligned} \theta &= \sec^{-1}3 \\ \sec\theta &= \sec\left(\sec^{-1}3\right) \\ \sec\theta &= 3 \\ \therefore \sin\left(2\sec^{-1}3\right) &= \sin(2\theta) = 2\sin\theta\cos\theta \\ &= 2\times\frac{\sqrt{8}}{3}\times\frac{1}{3} = \frac{4\sqrt{2}}{9} \end{aligned}$$

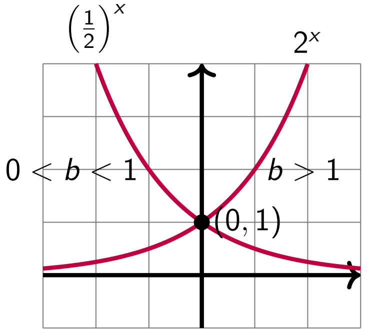


The Exponential Function

Definition 6

The exponential function to the base b is $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

$$\text{domain of } b^x = \mathbb{R}$$
$$\text{range of } b^x = (0, \infty)$$



Example 46

2^x , π^x , $(1/3)^x$, $\cdots$ are exponential

x^2 , x^π , $x^{1/3}$, $\cdots$ are NOT exponential

The Exponential Function

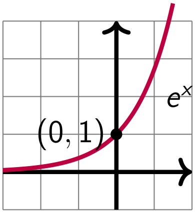
Example 47

Find the domain of the function $f(x) = 4^{\frac{x}{x^2-4}}$.

$$\text{dom}(f) = \mathbb{R} - \{x^2 - 4 = 0\} = \mathbb{R} - \{\pm 2\}$$

NOTE:

- The number $e \approx 2.7182818285\cdots$ is called the **Natural Number**.
- The exponential function $f(x) = e^x$ is called the **Natural Exponential Function**.



The Exponential Function: Properties

(1) $b^0 = 1$

Ex: $2^0 = 1$

(2) $b^x \cdot b^y = b^{x+y}$

Ex: $2^4 \cdot 2^3 = 2^7$

(3) $b^x \div b^y = b^{x-y}$

Ex: ${}^{3^5}/_{3^2} = 3^3$

(4) $b^{-x} = {}^1/_{b^x}$

Ex: $e^{-2} = {}^1/_{e^2}$

(5) $(b^x)^y = b^{xy}$

Ex: $\left(3^2\right)^3 = 3^6$

(6) $\sqrt[n]{b^m} = b^{m/n}$

Ex: $\sqrt[3]{2^6} = 2^{6/3} = 2^2$

(7) $(a \cdot b)^x = a^x \cdot b^x$

Ex: $(2e)^3 = 2^3 \cdot e^3 = 8e^3$

(8) $(a \div b)^x = a^x \div b^x$

Ex: $({}^5/_2)^2 = {}^{5^2}/_{2^2}$

(9) $b^x = b^y \Leftrightarrow x = y$

One-to-One Property

The Exponential Function: Properties

Example 48

Find the exact value:

(1) $(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$

(2) $9^{-1/2} = \dfrac{1}{9^{1/2}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

Example 49

Solve the equation $2^x = 64$.

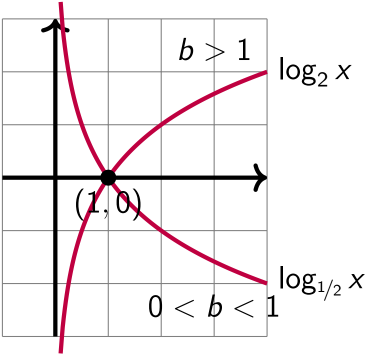
$$2^x = 64 \Rightarrow 2^x = 2^6 \Rightarrow x = 6$$

The Logarithmic Function

Definition 7

The logarithmic function to the base b is $f(x) = \log_b x$ where $b > 0$ and $b \neq 1$.

$$\text{domain of } \log_b x = (0, \infty)$$
$$\text{range of } \log_b x = \mathbb{R}$$



NOTE: The function $\log_e x$ is called the **Natural Logarithmic Function** and denoted by $\ln x$.

The Logarithmic Function

NOTE: $\text{dom}\left(\log_b g(x)\right) =$ All real numbers such that $g(x) > 0$ and defined.

Example 50

Find the domain of $f(x) = \ln\left(9 - x^2\right)$.

$$
\begin{aligned}
\text{dom}\ln\left(9 - x^2\right) = &\text{ All } x \in \mathbb{R} \text{ such that } 9 - x^2 > 0 \\
&\Rightarrow x^2 < 9 \\
&\Rightarrow \sqrt{x^2} < \sqrt{9} \\
&\Rightarrow |x| < 3 \\
&\Rightarrow -3 < x < 3 \\
&\Rightarrow x \in (-3, 3)
\end{aligned}
$$

The Logarithmic Function: Properties

(1) $\log_b 1 = 0$

Ex: $\log_{10} 1 = 0$

(2) $\log_b b = 1$

Ex: $\ln e = \log_e e = 1$

(3) $\log_b (x^n) = n \log_b x$

Ex: $\log_3 9 = \log_3 \left(3^2\right) = 2$

(4) $\log_b a = \dfrac{\ln a}{\ln b}$

Ex: $\log_4 3 = {}^{\ln 3}/_{\ln 4}$

(5) $\log_b(xy) = \log_b x + \log_b y$

(6) $\log_b \left({}^{x}/_{y}\right) = \log_b x - \log_b y$

(7) $\log_b x = \log_b y \Leftrightarrow x = y$

One-to-One Property

The Logarithmic Function: Properties

Example 51

Find the exact value.

(1) $\log_2 32 = \log_2 (2^5) = 5 \log_2 2 = 5$

(2) $\log_4 2 = \log_4 \sqrt{4} = \log_4 \left(4^{1/2}\right) = 1/2$

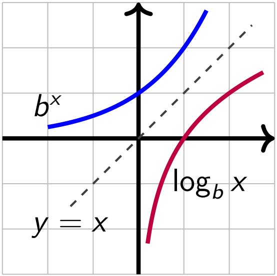
Example 52

Simplify the expression $\log_6 9 - \log_6 5 + \log_6 20$.

$$\begin{aligned}(\log_6 9 - \log_6 5) + \log_6 20 &= \log_6 \left(\frac{9}{5}\right) + \log_6 20 \\ &= \log_6 \left(\frac{9}{5} \times 20\right) \\ &= \log_6 36 = \log_6 \left(6^2\right) = 2\end{aligned}$$

Logarithmic & Exponential are Inverses

Function	Domain	Range
b^x	$\mathbb{R}$	$(0, \infty)$
$\log_b x$	$(0, \infty)$	$\mathbb{R}$



$$\log_b (b^x) = x \; ; \; x \in \mathbb{R}$$

$$b^{\log_b x} = x \; ; \; x \in (0, \infty)$$

Example 53

Find the exact value of $5^{2 \log_5 4}$.

$$5^{2 \log_5 4} = 5^{\log_5 \left(4^2\right)} = 4^2 = 16$$

Logarithmic & Exponential are Inverses

Example 54

Solve the following equations.

(1) $e^x = 7 \Rightarrow \ln\left(e^x\right) = \ln 7 \Rightarrow x = \ln 7$

(2) $\log_{10} x = -2 \Rightarrow 10^{\log_{10} x} = 10^{-2} \Rightarrow x = \dfrac{1}{10^2} = 0.01$

(3) $\ln x = 3 \Rightarrow e^{\ln x} = e^3 \Rightarrow x = e^3$

Logarithmic & Exponential are Inverses

Example 55

Solve the equation $e^{2x} + e^x - 6 = 0$.

$$
\begin{aligned}
e^{2x} + e^x - 6 = 0 &\Rightarrow (e^x)^2 + e^x - 6 = 0 \qquad (\text{Let } y = e^x) \\
&\Rightarrow y^2 + y - 6 = 0 \\
&\Rightarrow (y+3)(y-2) = 0 \\
\textbf{Solution (1)} &\Rightarrow y = -3 \\
&\Rightarrow e^x = -3 \qquad ✗ \\
\textbf{Solution (2)} &\Rightarrow y = 2 \\
&\Rightarrow e^x = 2 \qquad ✓ \\
&\Rightarrow x = \ln 2
\end{aligned}
$$

Logarithmic & Exponential are Inverses

Example 56

Solve the equation $\ln(x-2)+\ln(2x-3)=2\ln x$.

$$\begin{aligned}\ln\Big((x-2)(2x-3)\Big)=\ln\left(x^2\right) &\Rightarrow (x-2)(2x-3)=x^2\\ &\Rightarrow 2x^2-7x+6=x^2\\ &\Rightarrow x^2-7x+6=0\\ &\Rightarrow (x-6)(x-1)=0\\ &\Rightarrow x=6\ \checkmark \text{ or } x=1\ ✗\end{aligned}$$

Logarithmic & Exponential are Inverses

Example 57

True or False?

"The functions $\ln(x^2)$ and $2\ln x$ have the same domain !!"

$$\text{domain of } \ln\left(x^2\right) = \mathbb{R} - \{0\}$$
$$\text{domain of } 2\ln(x) = (0, \infty) \qquad \therefore \ \textbf{FALSE}$$

Example 58

Find the inverse function of $f(x) = \ln(x+3)$.

$$\begin{aligned} y = \ln(x+3) &\Rightarrow e^y = e^{\ln(x+3)} \\ &\Rightarrow x + 3 = e^y \\ &\Rightarrow x = e^y - 3 \qquad \therefore \ f^{-1}(x) = e^x - 3 \end{aligned}$$

Logarithmic & Exponential are Inverses

Example 59

Find the inverse function of $g(x) = e^{2x-1}$.

$$y = e^{2x-1} \Rightarrow \ln y = \ln\left(e^{2x-1}\right)$$
$$\Rightarrow 2x - 1 = \ln y$$
$$\Rightarrow x = \frac{1 + \ln y}{2} \qquad \therefore\ g^{-1}(x) = \frac{1 + \ln x}{2}$$

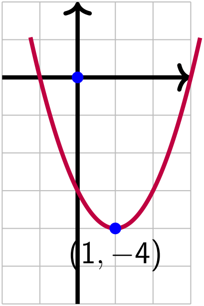
One More Example

Example 60

Let $f(x) = x^2 - 2x - 3$.

(1) Draw the graph of f by shifting the graph of x^2.

$$\left(x^2 - 2x\right) - 3 = \left(x^2 - 2x + 1\right) - 1 - 3$$
$$= (x-1)^2 - 4$$



(2) Find $f^{-1}(x)$.

$$y = (x-1)^2 - 4$$
$$x = \sqrt{y+4} + 1$$
$$f^{-1}(x) = \sqrt{x+4} + 1$$

(3) What is the range of f.

range of $f = [-4, \infty)$

$= \text{domain of } f^{-1}$