Lecture Notes for Calculus 101

Chapter 1 : Limits & Continuity

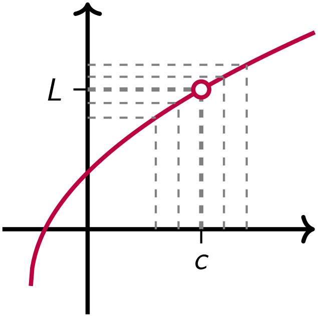
Feras Awad

Philadelphia University

The Idea of Limits

If $f(x)$ becomes close to a single L as x approaches c (from either sides), we say that "**the limit of** $f(x)$ **as** x **approaches to** c **is** L**.**"

$$\lim_{x \to c} f(x) = L$$



NOTE: The value of $f(c)$ has no effect on the value of $\lim_{x \to c} f(x)$.

Two Sided Limits

* The limit of $f(x)$ as x approaches "c" from left is denoted by $\lim_{x \to c^-} f(x)$
* The limit of $f(x)$ as x approaches "c" from right is denoted by $\lim_{x \to c^+} f(x)$

Theorem 1

*Let $L \in \mathbb{R}$. We say that $\lim_{x \to c} f(x)$ **exists** if and only if*

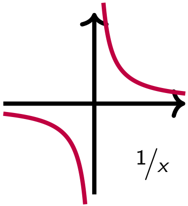
$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

NOTE: $x \to c^+$ means $x > c$

$x \to c^-$ means $x < c$

When the Limit Does Not Exist (**d.n.e**)?

(1) $\lim\limits_{x\to c^+} f(x) \neq \lim\limits_{x\to c^-} f(x)$

(2) 

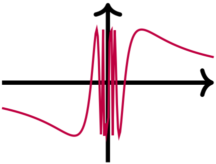
$$\lim_{x\to c^+} f(x) = \pm\infty$$

$$\lim_{x\to c^-} f(x) = \pm\infty$$

$$\lim_{x\to c} f(x) = \pm\infty$$

(3) $f(x)$ oscillating between two fixed values as x approaches to "c". For example,

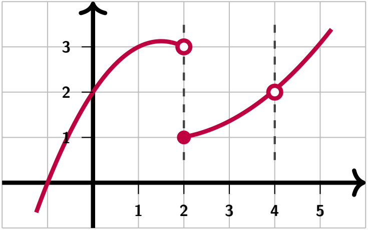
$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \textbf{d.n.e}$$



Limits from Function Graph

Example 1

From the given graph of $f(x)$, evaluate:



(1) $f(2) = 1$

(2) $\lim_{x \to 2^+} f(x) = 1$

(3) $\lim_{x \to 2^-} f(x) = 3$

(4) $\lim_{x \to 2} f(x)$ **d.n.e**

(5) $f(4)$ **undefined**

(6) $\lim_{x \to 4^+} f(x) = 2$

(7) $\lim_{x \to 4^-} f(x) = 2$

(8) $\lim_{x \to 4} f(x) = 2$

Laws of Limits

Theorem 2

Suppose that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ ***exist****, then:*

(1) $\lim_{x\to a}\left(f \overset{\bullet}{\underset{-}{+}} g\right) = \lim_{x\to a} f \overset{\bullet}{\underset{-}{+}} \lim_{x\to a} g$

(2) $\lim_{x\to a}\left(\dfrac{f}{g}\right) = \dfrac{\lim_{x\to a} f}{\lim_{x\to a} g} \quad ; \quad \lim_{x\to a} g \neq 0$

(3) $\lim_{x\to a} k = k \quad ; \quad k$ *is constant*

(4) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) \quad ; \quad k$ *is constant*

(5) $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n \quad ; \quad n$ *is positive integer*

(6) $\lim_{x\to a} \sqrt[n]{f} = \sqrt[n]{\lim_{x\to a} f} \quad ; \quad \lim_{x\to a} f(x) \geq 0$ *if* n *is even*

Laws of Limits

Example 2

Let $\lim_{x\to 1/2} f(x) = 2$ and $\lim_{x\to 1/2} g(x) = -1$, then

$$\lim_{x\to 1/2}\left(\frac{f(x)-2g(x)}{f(x)}\right)^2 = \left(\frac{2-2(-1)}{2}\right)^2 = 2^2 = 4$$

Example 3

Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{-1}{x}$. Note that $\lim_{x\to 0}\frac{1}{x}$ d.n.e and $\lim_{x\to 0}\frac{-1}{x}$ d.n.e, but:

$$\lim_{x\to 0}(f+g) = \lim_{x\to 0}\left(\frac{1}{x}+\frac{-1}{x}\right) = \lim_{x\to 0} 0 = 0$$

Direct Substitution Property

If $f(x)$ is almost any function (except piecewise) and $a \in \text{dom}(f)$, then $\lim_{x \to a} f(x) = f(a)$.

Example 4

Evaluate the following limits.

(1) $\lim_{x \to -2} x = -2$

(2) $\lim_{x \to 1} (x^2 + 3x - 1)^3 = 27$

(3) $\lim_{x \to 3} \frac{x+1}{x^2 - 4} = \frac{4}{5}$

(4) $\lim_{x \to -2} \frac{x+1}{x^2 - 4} = \frac{-1}{0}$ **d.n.e**

(5) $\lim_{x \to -9} \sqrt[3]{x+1} = \sqrt[3]{-8} = -2$

(6) $\lim_{x \to -1} \sqrt[3]{x+1} = \sqrt[3]{0} = 0$

(7) $\lim_{x \to 3} \sqrt{x+1} = 2$

(8) $\lim_{x \to -3} \sqrt{x+1} = \sqrt{-2}$ **d.n.e**

Direct Substitution Property

Example 5

Let $f(x) = \begin{cases} x+2 & : \quad x \leq -2 \\ x^2-5 & : \quad -2 < x < 3 \\ \sqrt{x+13} & : \quad x > 3 \end{cases}$. Find:

(1) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (x^2 - 5) = -4$

(2) $\lim\limits_{x \to -2} f(x)$ **d.n.e**

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} (x+2) = 0$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} \left(x^2 - 5\right) = -1$$

(3) $\lim\limits_{x \to 3} f(x) = 4$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} \left(x^2 - 5\right) = 4$$

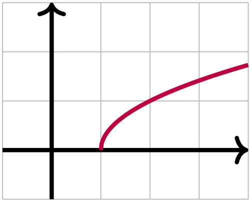
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} \sqrt{x+13} = 4$$

Direct Substitution Property

Example 6

Evaluate $\lim\limits_{t\to 1} \sqrt{t-1} = \sqrt{0} = 0$. **Wait !!**

$$\lim_{t\to 1^+} \sqrt{t-1} = 0$$
$$\lim_{t\to 1^-} \sqrt{t-1} \ \text{d.n.e}$$
$$\therefore \lim_{t\to 1} \sqrt{t-1} \ \text{d.n.e}$$



$\text{dom}\left(\sqrt{t-1}\right) = [1, \infty)$

Direct Substitution Property

Example 7

Evaluate $\lim_{t\to 1}\sqrt{t^2-2t+1}=\sqrt{0}=0$. **Wait !!**

$$\lim_{t\to 1^+}|t-1| = \lim_{t\to 1}(t-1) = 0$$

$$\lim_{t\to 1^-}|t-1| = \lim_{t\to 1}(1-t) = 0$$

$$\therefore \lim_{t\to 1}|t-1| = 0$$

$$\sqrt{t^2-2t+1} = \sqrt{(t-1)^2} = |t-1| = \begin{cases} t-1 & : t\geq 1 \\ -(t-1) & : t<1 \end{cases}$$

The Indeterminate Form $^0/_0$

Example 8

Evaluate $\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = \frac{0}{0}$ $\begin{cases} \textbf{Direct substitution fails.} \\ \textbf{The limit may exist or not.} \end{cases}$

$$\begin{aligned} \lim_{x \to -3} \frac{x^2 - 9}{x + 3} &= \lim_{x \to -3} \frac{(x-3)(x+3)}{x+3} \quad \text{(By Factoring)} \\ &= \lim_{x \to -3} (x - 3) \\ &= -6 \end{aligned}$$

NOTE: $a^2 - b^2 = (a - b)(a + b)$

The Indeterminate Form $0/0$

Example 9

Evaluate $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4x+4} = \frac{0}{0}$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{(x-2)(x-2)}$$

$$= \lim_{x\to 2} \frac{x+3}{x-2}$$

$$= \frac{5}{0} \quad \textbf{d.n.e}$$

The Indeterminate Form $^0/_0$

Example 10

Evaluate $\lim_{t\to 2} \frac{t^3 - 8}{t - 2} = \frac{0}{0}$

$$= \lim_{t\to 2} \frac{(t-2)\left(t^2 + 2t + 4\right)}{t-2}$$

$$= \lim_{t\to 2} \left(t^2 + 2t + 4\right)$$

$$= 12$$

NOTE: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

The Indeterminate Form $^0/_0$

Example 11

Evaluate $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = \dfrac{0}{0}$ (By Conjugate)

$$= \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4} \frac{(x-4)\left(\sqrt{x}+2\right)}{x-4}$$

$$= \lim_{x\to 4} \left(\sqrt{x}+2\right) = 4$$

NOTE: $a^2 - b^2 = (a-b)(a+b)$

$$x - 4 = \left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)$$

The Indeterminate Form $^0/_0$

Example 12

Evaluate $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{0}{0}$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \\
&= \lim_{x\to 0} \frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} \\
&= \frac{1}{4}
\end{aligned}
$$

The Indeterminate Form $0/0$

Example 13

Evaluate $\lim_{x\to -8} \dfrac{\sqrt[3]{x}+2}{x+8} = \dfrac{0}{0}$ $\begin{cases} \text{By substituting } y = \sqrt[3]{x} \Rightarrow x = y^3 \\ \text{If } x \to -8, \text{ then } y \to \sqrt[3]{-8} = -2 \end{cases}$

$$= \lim_{y\to -2} \frac{y+2}{y^3+8}$$

$$= \lim_{y\to -2} \frac{y+2}{(y+2)\left(y^2-2y+4\right)}$$

$$= \lim_{y\to -2} \frac{1}{y^2-2y+4}$$

$$= \frac{1}{12}$$

The Indeterminate Form $^0/_0$

Example 14

Evaluate $\lim_{x\to 5} \frac{\frac{1}{x} - \frac{1}{5}}{x-5} = \frac{0}{0}$

$$= \lim_{x\to 5} \frac{\left(\frac{5-x}{5x}\right)}{x-5}$$

$$= \lim_{x\to 5} \left(\frac{5-x}{5x} \cdot \frac{1}{x-5}\right)$$

$$= \lim_{x\to 5} \frac{-1}{5x}$$

$$= \frac{-1}{25}$$

The Indeterminate Form $0/0$

Example 15

Evaluate $\lim\limits_{x\to 3} \dfrac{|x-3|}{x^2-9} = \dfrac{0}{0}$

$$|x-3| = \begin{cases} x-3 & : x \geq 3 \\ -(x-3) & : x < 3 \end{cases}$$

$$\frac{|x-3|}{x^2-9} = \begin{cases} \frac{x-3}{x^2-9} & : x > 3 \\ \frac{-(x-3)}{x^2-9} & : x < 3 \end{cases}$$

$$= \begin{cases} \frac{1}{x+3} & : x > 3 \\ \frac{-1}{x+3} & : x < 3 \end{cases}$$

$$\lim_{x\to 3^+} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^+} \frac{1}{x+3} = \frac{1}{6}$$

$$\lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{-1}{x+3} = \frac{-1}{6}$$

$$\therefore \lim_{x\to 3} \frac{|x-3|}{x^2-9} \quad \textbf{d.n.e}$$

The Indeterminate Form $^0/_0$

Example 16

If $\underbrace{\lim_{x\to 1} \frac{f(x)}{x-1} = 5}$, find $\underbrace{\lim_{x\to 1}\left(2f(x)+x^2\right)}$

Since $5 \in \mathbb{R}$

then the limit exists

Since $\lim_{x\to 1}(x-1) = 0$

then $\lim_{x\to 1} f(x) = 0$

$$= 2\lim_{x\to 1} f(x) + \lim_{x\to 1} x^2$$

$$= 2(0) + 1$$

$$= 1$$

The Indeterminate Form $^0/_0$

Example 17

Evaluate $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \dfrac{0}{0}$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \cdot \frac{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}+1}{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}+1}$$

$$= \lim_{x\to 1} \frac{x-1}{x-1} \cdot \frac{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}+1}{\sqrt{x}+1} = \frac{3}{2}$$

NOTE: $a^3 - b^3 = (a-b)\left(a^2+ab+b^2\right)$

$$x - 1 = \left(\sqrt[3]{x}-1\right)\left(\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}+1\right)$$

The Indeterminate Form 0/0

Example 18

Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & : x \neq 2 \\ 5 & : x = 2 \end{cases}$. Find:

(1) $f(2) = 5$

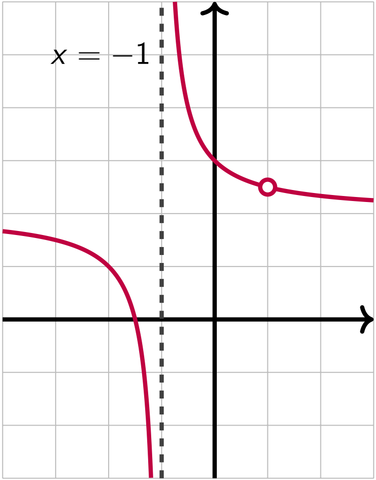
(2) $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

Infinite Limits and Vertical Asymptotes

The figure shows the graph of
$f(x) = \dfrac{2x^2 + x - 3}{x^2 - 1}$.

$$\lim_{x \to -1^-} f(x) = -\infty$$
$$\lim_{x \to -1^+} f(x) = \infty$$

$\therefore\ x = -1$ is **Vertical Asymptote**



Infinite Limits and Vertical Asymptotes

Definition 1

We say that $x = a$ is vertical asymptote of $f(x)$ if one of the following holds:

$$\lim_{x\to a^+} f(x) = \pm\infty$$
$$\text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$
$$\text{or} \quad \lim_{x\to a} f(x) = \pm\infty$$

Example 19

Find the vertical asymptotes for $f(x) = \dfrac{x-1}{x^2-1}$.

Since $f(x) = \dfrac{x-1}{(x-1)(x+1)} = \dfrac{1}{x+1}$, then $x = -1$ is **V.A.**

Infinite Limits and Vertical Asymptotes

Example 20

Find the vertical asymptotes for $f(x) = \dfrac{2x^3 - 5x + 7}{x^2 + 4x - 5}$.

$$\begin{aligned} \text{denominator} = 0 &\Rightarrow x^2 + 4x - 5 = 0 \\ &\Rightarrow (x-1)(x+5) = 0 \\ &\Rightarrow x = 1\,,\ x = -5 \end{aligned}$$

Now, substitute $x = 1$ in the numerator $\Rightarrow 4 \neq 0$

$x = -5$ in the numerator $\Rightarrow -218 \neq 0$

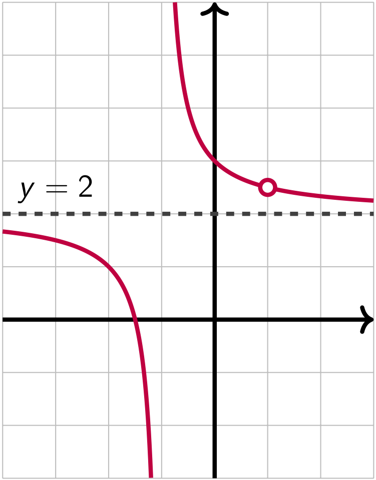
$\therefore\ x = 1$ and $x = -5$ are **V.As**

Limits at Infinity and Horizontal Asymptote

The figure shows the graph of
$f(x) = \dfrac{2x^2 + x - 3}{x^2 - 1}$.

$$\lim_{x \to \infty} f(x) = 2$$
$$\lim_{x \to -\infty} f(x) = 2$$

$\therefore\ y = 2$ is **Horizontal Asymptote**



Limits at Infinity and Horizontal Asymptote

Definition 2

We say that $y = b$ is a horizontal asymptote of $f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

NOTE:

(1) $\lim_{x\to\pm\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x\to\pm\infty} a_n x^n$

(2) $\lim_{x\to\pm\infty} k = k$ where k is constant

(3) $$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \begin{cases} \pm\infty & : n > m \\ 0 & : n < m \\ {}^{a_n}\!/_{b_m} & : n = m \end{cases}$$

Limits at Infinity and Horizontal Asymptote

Example 21

Find the horizontal asymptotes for $f(x) = \dfrac{x-1}{x^2-1}$.

$$\begin{aligned}\lim_{x\to\infty}\frac{x-1}{x^2-1} &= \lim_{x\to\infty}\frac{x}{x^2}\\ &= \lim_{x\to\infty}\frac{1}{x}\\ &= \frac{1}{\infty}\\ &= 0\\ \therefore\ y &= 0 \text{ is } \textbf{H.A}\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty}\frac{x-1}{x^2-1} &= \lim_{x\to-\infty}\frac{x}{x^2}\\ &= \lim_{x\to-\infty}\frac{1}{x}\\ &= \frac{1}{-\infty}\\ &= 0\\ \therefore\ y &= 0 \text{ is } \textbf{H.A}\end{aligned}$$

Limits at Infinity and Horizontal Asymptote

Example 22

Find the horizontal asymptotes for $f(x) = \dfrac{2x^3 - 5x + 7}{4x - 3x^2}$.

$$\lim_{x\to\infty} \frac{2x^3 - 5x + 7}{4x - 3x^2} = \lim_{x\to\infty} \frac{2x^3}{-3x^2} = \lim_{x\to\infty} \frac{-2x}{3} = -\infty$$

$\therefore$ There is **NO H.A**

$$\lim_{x\to-\infty} \frac{2x^3 - 5x + 7}{4x - 3x^2} = \lim_{x\to-\infty} \frac{2x^3}{-3x^2} = \lim_{x\to-\infty} \frac{-2x}{3} = \infty$$

$\therefore$ There is **NO H.A**

Limits at Infinity and Horizontal Asymptote

Example 23

Find the horizontal asymptotes for $f(x) = \dfrac{(2x^2-1)(x+1)}{-4x^3+2x^2-x+1}$.

$$\lim_{x\to\infty} \frac{(2x^2-1)(x+1)}{-4x^3+2x^2-x+1} = \lim_{x\to\infty} \frac{(2x^2)(x)}{-4x^3} = \lim_{x\to\infty} \frac{2x^3}{-4x^3} = -\frac{1}{2}$$

$$\therefore\ y = {}^{-1}\!/_2 \text{ is } \mathbf{H.A}$$

$$\lim_{x\to-\infty} \frac{(2x^2-1)(x+1)}{-4x^3+2x^2-x+1} = \lim_{x\to-\infty} \frac{(2x^2)(x)}{-4x^3} = \lim_{x\to-\infty} \frac{2x^3}{-4x^3} = -\frac{1}{2}$$

$$\therefore\ y = {}^{-1}\!/_2 \text{ is } \mathbf{H.A}$$

Limits at Infinity and Horizontal Asymptote

Example 24

Find the horizontal asymptotes for $f(x) = \dfrac{\sqrt{4x^2+1}}{3x-1}$.

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{3x-1} = \lim_{x\to\infty} \frac{\sqrt{4x^2}}{3x} = \lim_{x\to\infty} \frac{2|x|}{3x} = \lim_{x\to\infty} \frac{2x}{3x} = \frac{2}{3}$$

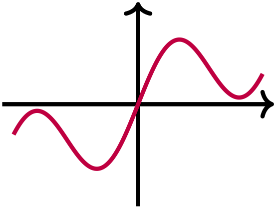
$\therefore\ y = {}^{2}\!/_{3}$ is **H.A**

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{3x-1} = \lim_{x\to-\infty} \frac{\sqrt{4x^2}}{3x} = \lim_{x\to-\infty} \frac{2|x|}{3x} = \lim_{x\to-\infty} \frac{-2x}{3x} = \frac{-2}{3}$$

$\therefore\ y = {}^{-2}\!/_{3}$ is **H.A**

Continuous Functions

Simply, a function is continuous if it does not have any "breaks" or "holes" in its graph.



Definition 3

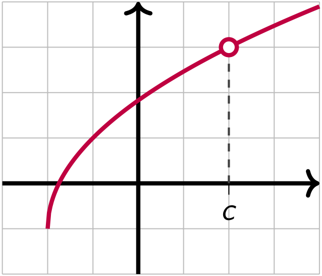
A function $f(x)$ is continuous at $x = c$ if all the following three conditions are hold:

(1) $f(c)$ is defined

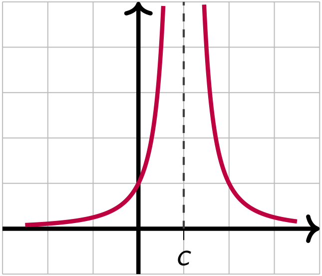
(2) $\lim_{x \to c} f(x)$ exists

(3) $f(c) = \lim_{x \to c} f(x)$

When $f(x)$ is discontinuous at $x = c$?



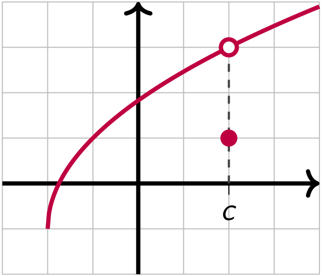
$f(c)$ is not defined



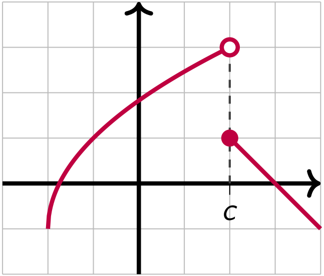
$f(c)$ is not defined

$\lim_{x \to c} f(x)$ d.n.e

When $f(x)$ is discontinuous at $x = c$?



$$f(c) \neq \lim_{x \to c} f(x)$$



$$\lim_{x \to c} f(x) \text{ d.n.e}$$

When $f(x)$ is discontinuous at $x = c$?

Example 25

Determine whether $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & : x \neq 2 \\ 4 & : x = 2 \end{cases}$ is continuous at $x = 2$?

$$
\begin{array}{ll}
\checkmark & f(2) = 4 \\
\checkmark & \displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4 \\
\checkmark & f(2) = \displaystyle\lim_{x \to 2} f(x) \\
\therefore & f(x) \text{ is continuous at } x = 2
\end{array}
$$

Continuity on Intervals

* $f(x)$ is continuous on (a, b) or $(-\infty, \infty)$ if f is continuous at each $x = c$ in the interval.

* $f(x)$ is continuous on $[a, b]$ if
  - f is continuous on (a, b)
  - f is continuous at $x = a$ from **right**.
    $\lim_{x \to a^+} f(x) = f(a)$
  - f is continuous at $x = b$ from **left**.
    $\lim_{x \to b^-} f(x) = f(b)$

NOTE: In general, every function (except piecewise) is continuous on its domain.

Continuity on Intervals

Example 26

Let $f(x) = \dfrac{x^2 - 9}{x^2 - 5x + 6}$. Where f is discontinuous?

$$\begin{aligned} f \text{ is discontinuous} &\Leftrightarrow x^2 - 5x + 6 = 0 \\ &\Leftrightarrow (x-2)(x-3) = 0 \Leftrightarrow x = 2\,,\ x = 3 \end{aligned}$$

Example 27

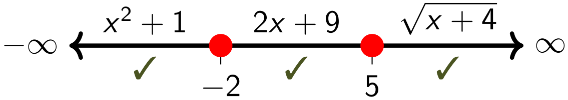
Let $g(x) = \sqrt{4 - x^2}$. Where g is continuous?

$$\begin{aligned} g \text{ is continuous} &\Leftrightarrow 4 - x^2 \geq 0 \\ &\Leftrightarrow x^2 \leq 4 \Leftrightarrow |x| \leq 2 \Leftrightarrow -2 \leq x \leq 2 \end{aligned}$$

Continuity on Intervals

Example 28

Let $g(x) = \begin{cases} x^2+1 & : x \le -2 \\ 2x+9 & : -2 < x < 5 \\ \sqrt{x+4} & : x \ge 5 \end{cases}$. Where g is discontinuous?



$x=-2 \quad g(-2)=5 \quad \lim_{x\to -2^-} g(x)=5 \quad \lim_{x\to -2^+} g(x)=5$ ✓

$x=5 \quad g(5)=3 \quad \lim_{x\to 5^-} g(x)=19 \quad \lim_{x\to 5^+} g(x)=3$ ✗

$\therefore \quad g(x)$ is discontinuous at $x=5$.

Continuity on Intervals

Example 29

Find c such that $f(x) = \begin{cases} \dfrac{x^2-9}{x+3} & : x \neq -3 \\ c^3 & : x = -3 \end{cases}$ is continuous every where.

$$\begin{aligned} f \text{ continuous every where} &\Rightarrow f \text{ continuous at } x = -3 \\ &\Rightarrow f(-3) = \lim_{x \to -3} f(x) \\ &\Rightarrow c^3 = \lim_{x \to -3} \frac{x^2-9}{x+3} \\ &\Rightarrow c^3 = -6 \Rightarrow c = \sqrt[3]{-6} = -\sqrt[3]{6} \end{aligned}$$

Some Theorems

Theorem 3

Suppose f and g are continuous at $x = c$, then:

(1) $f \pm g$ *is continuous at* $x = c$.

(2) $\dfrac{f}{g}$ *is continuous at* $x = c$ *if* $g(c) \neq 0$.

Theorem 4

If g is continuous at $x = c$, and f is continuous at $g(c)$, then $f \circ g$ is continuous at $x = c$.

Some Theorems

Theorem 5

The absolute value of continuous function is continuous.

For example, $g(x) = |3x^2 - 4x + 1|$ is continuous on $\mathbb{R}$.

Theorem 6

If $\lim_{x \to c} g(x) = L$ *(exists) and* $f(x)$ *is continuous at* $x = L$*, then*
$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$

Example 30

$$\lim_{x \to 2} \log_2 \left(\frac{x^2 - 4}{x - 2}\right) = \log_2 \left(\lim_{x \to 2} \frac{x^2 - 4}{x - 2}\right) = \log_2 4 = 2$$

Exercises

Exercise 1

(1) *Find k such that* $f(x) = \begin{cases} 7x - 1 & : x \leq 1 \\ kx^2 & : x > 1 \end{cases}$ *is continuous on* $\mathbb{R}$.

(2) *Find the interval of continuity of* $g(x) = \dfrac{\ln\left(4 - x^2\right)}{x}$

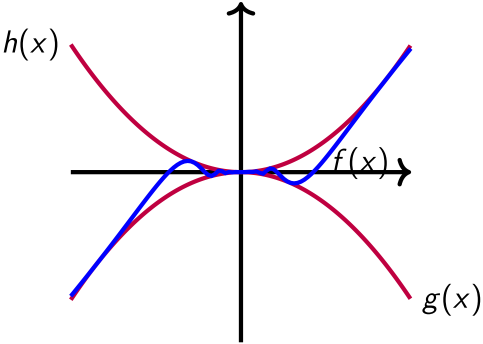
(3) *Find the vertical and horizontal asymptotes of* $f(x) = \dfrac{5x^3 + 6x}{x^3 - 4x}$

Squeeze Theorem

Let f, g and h be functions such that

$$g(x) \leq f(x) \leq h(x) \; ; \; \forall x \in \text{some interval}$$

$$\text{If } \lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L, \text{ then } \lim_{x\to c} f(x) = L.$$



Squeeze Theorem

Example 31

Given that $f(x)$ satisfies the inequality

$$1 - x^2 \leq f(x) \leq \cos x \; ; \quad \text{for all } x \in \left(-\pi/2, \pi/2\right)$$

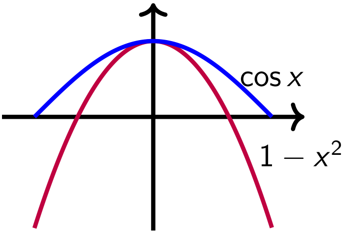
Find $\lim_{x \to 0} f(x)$

Since $\lim_{x \to 0} \left(1 - x^2\right) = 1$

$\lim_{x \to 0} \cos x = 1$

then by Squeezing Theorem

$\lim_{x \to 0} f(x) = 1$



Squeeze Theorem

Example 32

Use Squeeze Theorem to evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

$$
\begin{aligned}
-1 \le \sin x \le 1 &\Rightarrow -1 \le \sin\left(\frac{1}{x}\right) \le 1 \\
&\Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2
\end{aligned}
$$

Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$, then $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

The Limit of $\sin x/x$

Theorem 7

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\tan x}{x}=1 \qquad \lim_{x\to 0}\frac{\sin x}{\tan x}=1$$

$$\lim_{x\to 0}\frac{x}{\sin x}=1 \qquad \lim_{x\to 0}\frac{x}{\tan x}=1 \qquad \lim_{x\to 0}\frac{\tan x}{\sin x}=1$$

$$\lim_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b} \qquad \lim_{x\to 0}\frac{\tan(ax)}{bx}=\frac{a}{b} \qquad \lim_{x\to 0}\frac{\sin(ax)}{\tan(bx)}=\frac{a}{b}$$

Example 33

$$\lim_{x\to 0}\frac{\tan(2x)}{\sin(3x)}=\frac{2}{3}$$

The Limit of $\sin x/x$

Example 34

$$\lim_{x\to 0}\frac{\sin^2(2x)}{3x^2}=\frac{1}{3}\lim_{x\to 0}\frac{\sin^2(2x)}{x^2}=\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin(2x)}{x}\right)^2=\frac{2^2}{3}=\frac{4}{3}.$$

Example 35

$$\lim_{x\to 0}\frac{\tan\left(2x^2\right)}{3x}=\lim_{x\to 0}\frac{\tan\left(2x^2\right)}{3x}\cdot\frac{x}{x}=\lim_{x\to 0}x\cdot\frac{\tan\left(2x^2\right)}{3x^2}$$

$$=\left(\lim_{x\to 0}x\right)\left(\lim_{x\to 0}\frac{\tan\left(2x^2\right)}{3x^2}\right)\begin{cases}\text{Let } y=x^2\\ \text{If } x\to 0 \text{ then } y\to 0\end{cases}$$

$$=\left(\lim_{x\to 0}x\right)\left(\lim_{y\to 0}\frac{\tan(2y)}{3y}\right)=0\times\frac{2}{3}=0$$

The Limit of $\sin x/x$

Example 36

$$\begin{aligned}
\lim_{x\to 5} \frac{x-5}{\sin\left(x^2-25\right)} &= \lim_{x\to 5} \frac{x-5}{\sin\left(x^2-25\right)} \cdot \frac{x+5}{x+5} \\
&= \lim_{x\to 5} \frac{1}{x+5} \cdot \frac{x^2-25}{\sin\left(x^2-25\right)} \\
&= \left(\lim_{x\to 5} \frac{1}{x+5}\right) \underbrace{\left(\lim_{x\to 5} \frac{x^2-25}{\sin\left(x^2-25\right)}\right)}_{\substack{\text{Let } y=x^2-25 \\ \text{If } x\to 5, \text{ then } y\to 0}} \\
&= \left(\lim_{x\to 5} \frac{1}{x+5}\right) \left(\lim_{y\to 0} \frac{y}{\sin\left(y\right)}\right) = \frac{1}{10} \times 1 = \frac{1}{10}
\end{aligned}$$

The Limit of $\sin x/x$

Example 37

$$\lim_{x\to 0}\frac{\sin(3x)+\tan(4x)}{x^2+2x}=\lim_{x\to 0}\frac{\dfrac{\sin(3x)}{x}+\dfrac{\tan(4x)}{x}}{x+2}=\frac{3+4}{0+2}=\frac{7}{2}$$

Example 38

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)=1\times\frac{0}{1+1}=0$$

The Limit of $\sin x/x$

Example 39

Find a nonzero constant k that makes the function
$f(x) = \begin{cases} \dfrac{\sin(kx)}{x} & : x < 0 \\ 3x + 2k^2 & : x \geq 0 \end{cases}$ continuous at $x = 0$.

$$\begin{aligned} f \text{ continuous at } x = 0 &\Leftrightarrow \lim_{x\to 0} f(x) \text{ exists} \\ &\Leftrightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \\ &\Leftrightarrow \lim_{x\to 0^+} \left(3x + 2k^2\right) = \lim_{x\to 0^-} \frac{\sin(kx)}{x} \\ &\Leftrightarrow 2k^2 = k \Leftrightarrow 2k^2 - k = 0 \\ &\Leftrightarrow k(2k-1) = 0 \Leftrightarrow k = 0 \text{ ✗ or } k = 1/2 \text{ ✓} \end{aligned}$$

Exercises

Exercise 2

(1) *Suppose that* $4 - x^2 \leq \dfrac{f(x)}{x-2} \leq e^x + e^{-x} - 6$. *Use Squeeze Theorem to evaluate* $\lim_{x \to 0} f(x)$.

(2) *Suppose that* $x^2 - \dfrac{x^4}{3} \leq f(x) \leq x^2$ *for all* $x \in \mathbb{R}$. *Use Squeeze Theorem to evaluate* $\lim_{x \to 0} \dfrac{f(x)}{x^2}$.

Exercises

Exercise 3

Evaluate the following limits

(1) $\lim\limits_{x \to 1} \dfrac{\sin(x-1)}{x^2+2x-3}$

(2) $\lim\limits_{\theta \to 0} \dfrac{2\sin^2\theta}{\theta\cos\theta}$

(3) $\lim\limits_{x \to 0} \dfrac{2-\cos(3x)-\cos(4x)}{x}$