Lecture Notes for Calculus 101

Chapter 2 : Differentiation & its Applications

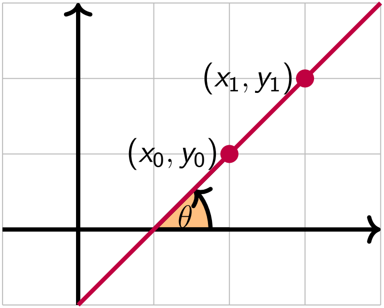
Feras Awad

Philadelphia University

Equation of Line (Point-Slope Formula)

The slope $m = \tan\theta$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$



NOTE: The equation of the line with slope m and passes through the point (x_0, y_0) is given by

$$y - y_0 = m\,(x - x_0)$$

Equation of Line (Point-Slope Formula)

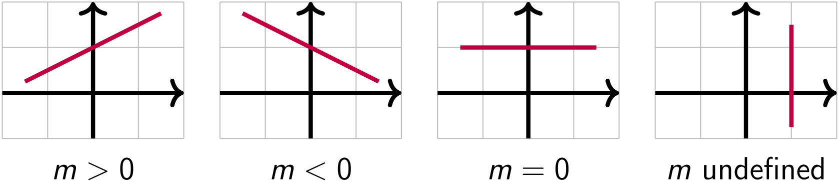
Example 1

Find the equation of the line passes through the points $(1,4)$ and $(2,7)$.

$$\begin{aligned} m &= \frac{7-4}{2-1} = 3 \\ y-4 &= 3(x-1) \\ y-4 &= 3x-3 \\ y &= 3x+1 \end{aligned}$$

NOTE: The equation of the form $y = mx + b$ is called the **slope-intercept** formula, where m is the slope of the line, and b is the $y-$intercept.

The Relations Between Lines



Let ℓ_1 be a line with slope m_1, and ℓ_2 be a line with slope m_2. Then

(1) ℓ_1 and ℓ_2 are parallel $\Leftrightarrow m_1 = m_2$.

(2) ℓ_1 and ℓ_2 are perpendicular $\Leftrightarrow m_1 \times m_2 = -1$.

(3) Otherwise, ℓ_1 and ℓ_2 intersects at some point.

The Relations Between Lines

Example 2

(1) If the line $y = mx - 5$ is *parallel* to the line $6x - 3y = 12$, then $m =$

(A) 6 (B) 2 ✓ (C) -2 (D) $-1/2$

(2) If the line $y = mx - 5$ is *perpendicular* to the line $6x - 3y = 12$, then $m =$

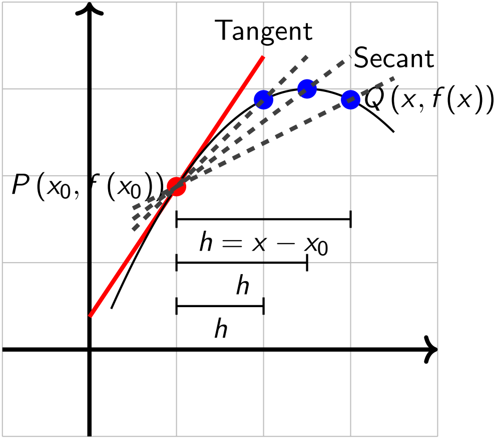
(A) 6 (B) 2 (C) -2 (D) $-1/2$ ✓

Write the line $6x - 3y = 12$ in standard form as $y = 2x - 4$.

The Equation of the Tangent Line

$$m_{\text{sec}} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\begin{aligned} m_{\tan} &= \lim_{x \to x_0} m_{\text{sec}} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= f'(x_0) \end{aligned}$$



The Equation of the Tangent Line

Example 3

Find the equation of the tangent line of $f(x) = x^2$ at $x_0 = 3$.

$$\text{Point: } (x_0, f(x_0)) = (3, f(3)) = (3, 9)$$

$$\text{Slope: } m_{\tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

$$\begin{aligned} \text{Equation: } y - f(x_0) &= m_{\tan}(x - x_0) \\ y - f(3) &= 6(x - 3) \\ y - 9 &= 6x - 18 \\ y &= 6x - 9 \end{aligned}$$

The Definition of the Derivative

Definition 1

The 1st order derivative of $y = f(x)$ at $x = c$ is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

NOTE: The first order derivative of the function $y = f(x)$ is denoted by

$$y' \quad , \quad f'(x) \quad , \quad \frac{dy}{dx} \quad , \quad \frac{df}{dx} \quad , \quad \frac{d}{dx}f(x) \quad , \quad Dy$$

The Definition of the Derivative

Example 4

If $f(x) = x^3$, find $f'(c)$.

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{x^3 - c^3}{x - c} \\ &= \lim_{x \to c} \frac{(x - c)\left(x^2 + cx + c^2\right)}{x - c} \\ &= \lim_{x \to c} \left(x^2 + cx + c^2\right) = 3c^2 \end{aligned}$$

When $f(x)$ is Differentiable at $x = c$?

Definition 2

$f(x)$ is differentiable at $x = c$ if $f'(c)$ exists.

NOTE: $f'(c)$ exists $\Leftrightarrow \lim\limits_{x\to c} \dfrac{f(x) - f(x)}{x - c}$ exists

$$\Leftrightarrow \lim_{x\to c^+} \frac{f(x) - f(x)}{x - c} = \lim_{x\to c^-} \frac{f(x) - f(x)}{x - c}$$

$$\Leftrightarrow f'_+(c) = f'_-(c)$$

When $f(x)$ is Differentiable at $x = c$?

Example 5

Show that $f(x) = |x|$ is **NOT** differentiable at $x = 0$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}$$

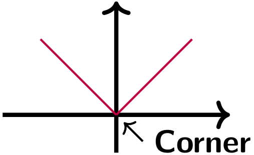
$$f'_+(0) = \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{|x|}{x} = -1$$

$\therefore f'(0)$ **d.n.e**

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

$$\frac{|x|}{x} = \begin{cases} 1 & : x > 0 \\ -1 & : x < 0 \end{cases}$$



The Relation Between Continuity & Differentiablity

Theorem 1

(1) *If $f(x)$ is **discontinuous** at $x = c$, then $f'(c)$ **d.n.e***

(2) *If $f'(c)$ **exists**, then f is **continuous** at $x = c$*

Example 6

$f(x) = \dfrac{1}{x}$ is discontinuous at $x = 0 \Rightarrow f'(0)$ d.n.e

NOTE:

(1) If f is continuous at $x = c$, then $f'(c)$ may exist or may not.

(2) If $f'(c)$ d.n.e, then f may continuous at $x = c$ or may not.

Rules of Differentiation & Chain Rule

Rule [1]: $\frac{d}{dx}\Big(\text{constant}\Big) = 0$

For example, $\frac{d}{dx}\left(1/2\right) = 0$, $\frac{d}{dx}\left(\pi^2\right) = 0$

Rule [2]: $\frac{d}{dx}\Big(kf(x)\Big) = kf'(x) \quad ; \quad k \text{ is constant}$

Rule [3]: $\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = f'(x) \pm g'(x)$

Rule [4]: $\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

$$\frac{d}{dx}\Big(f \cdot g \cdot h\Big) = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

$$\vdots$$

Rules of Differentiation & Chain Rule

Rule [5]: $$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$$\frac{d}{dx}\left(\frac{c}{g}\right) = \frac{-c \cdot g'}{g^2} \quad ; \quad c \text{ is constant}$$

Rule [6]: Chain Rule

$$\frac{d}{dx}\Big((f \circ g)(x)\Big) = \frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

Rule [7]: $$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}\left((g(x))^n\right) = n \cdot (g(x))^{n-1} \cdot g'(x)$$

Rules of Differentiation & Chain Rule

Example 7

(1) $\dfrac{d}{dx}\left(x\right) = 1 \cdot x^0 = 1$

(2) $\dfrac{d}{dx}(2x+3) = 2\dfrac{d}{dx}(x) + \dfrac{d}{dx}(3) = 2$

(3) $\dfrac{d}{dx}\left(4x^2\right) = 4\dfrac{d}{dx}\left(x^2\right) = (4)(2x) = 8x$

(4)
$$\begin{aligned}\frac{d}{dx}\left(\left(x^2-3x\right)^3\right) &= 3\cdot\left(x^2-3x\right)^2\cdot\frac{d}{dx}\left(x^2-3x\right)\\ &= 3\cdot\left(x^2-3x\right)^2\cdot(2x-3)\end{aligned}$$

Rules of Differentiation & Chain Rule

Example 8

(1) $$\begin{aligned}\frac{d}{dx}\left(x(1-x)^{100}\right) &= x\cdot\frac{d}{dx}\left((1-x)^{100}\right)+\frac{d}{dx}(x)\cdot(1-x)^{100}\\ &= x\cdot 100(1-x)^{99}(-1)+(1-x)^{100}\\ &= -100x(1-x)^{99}+(1-x)^{100}\\ &= (1-x)^{99}(1-101x)\end{aligned}$$

(2) $$\frac{d}{dx}\left(\frac{3}{x}\right)=3\frac{d}{dx}\left(x^{-1}\right)=3\cdot(-1)x^{-2}=\frac{-3}{x^2}$$

$$\frac{d}{dx}\left(\frac{3}{x}\right)=\frac{-3\cdot(x)'}{x^2}=\frac{-3}{x^2}$$

Rules of Differentiation & Chain Rule

Example 9

(1) $$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)\cdot(x)' - x\cdot(x^2+1)'}{(x^2+1)^2}$$
$$= \frac{(x^2+1)\cdot(1) - x\cdot(2x)}{(x^2+1)^2}$$
$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

(2) $$\frac{d}{dt}\Big((t-1)(t+1)\Big) = \frac{d}{dt}\left(t^2-1\right) = 2t$$

(3) $$\frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}\cdot x^{-1/3} = \frac{2}{3}\cdot\frac{1}{x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

Rules of Differentiation & Chain Rule

NOTE: $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$; $\frac{d}{dx}\left(\sqrt{g(x)}\right) = \frac{g'(x)}{2\sqrt{g(x)}}$

Example 10

$$\frac{d}{dx}\left(\sqrt{x^2-1}\right) = \frac{\frac{d}{dx}(x^2-1)}{2\sqrt{x^2-1}} = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

Example 11

$$\frac{d}{dx}\left(\frac{x^2+\sqrt{x}}{x}\right) = \frac{d}{dx}\left(\frac{x^2}{x}+\frac{x^{1/2}}{x}\right) = \frac{d}{dx}\left(x+x^{-1/2}\right) = 1-\frac{1}{2}x^{-3/2}$$
$$= 1-\frac{1}{2\sqrt{x^3}}$$

Rules of Differentiation & Chain Rule

Rule [8]: $\dfrac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} \cdot \ln b \cdot g'(x)$

Example 12

(1) $\dfrac{d}{dx}(e^x) = e^x \cdot \ln e \cdot \dfrac{d}{dx}(x) = e^x$

(2) $\dfrac{d}{dx}\left(3^{x^2+1}\right) = 3^{x^2+1} \cdot \ln 3 \cdot \dfrac{d}{dx}(x^2+1) = 2\ln 3 \cdot x \cdot 3^{x^2+1}$

(3) $$\begin{aligned}\frac{d}{dx}(x \cdot 2^x) &= x\frac{d}{dx}(2^x) + 2^x\frac{d}{dx}(x) \\ &= x \cdot 2^x \cdot \ln 2 \cdot 1 + 2^x \cdot 1 = 2^x(x\ln 2 + 1)\end{aligned}$$

Rules of Differentiation & Chain Rule

Example 13

(1) $$\frac{dx}{dx}\left(\sqrt{e^x}\right) = \frac{e^x}{2\sqrt{e^x}} = \frac{e^x}{2\left(e^x\right)^{1/2}} = \frac{e^x}{2e^{x/2}} = \frac{1}{2}e^{x/2} = \frac{1}{2}\sqrt{e^x}$$

OR $$\frac{dx}{dx}\left(\sqrt{e^x}\right) = \frac{d}{dx}\left(e^{x/2}\right) = e^{x/2}\cdot\frac{d}{dx}\left({}^{x}\!/_{2}\right) = \frac{1}{2}e^{x/2} = \frac{1}{2}\sqrt{e^x}$$

(2) $$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x\cdot 1 - x\cdot e^x}{\left(e^x\right)^2} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$

OR $$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{d}{dx}\left(xe^{-x}\right) = x\cdot\frac{d}{dx}\left(e^{-x}\right) + e^{-x}\cdot\frac{d}{dx}(x)$$

$$= -xe^{-x} + e^{-x} = e^{-x}(-x+1) = \frac{1-x}{e^x}$$

Rules of Differentiation & Chain Rule

Rule [9]: $\dfrac{d}{dx}\left(\log_b g(x)\right) = \dfrac{g'(x)}{g(x)\ln b}$

Example 14

(1) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x \ln e} = \dfrac{1}{x}$

(2) $\dfrac{d}{dx}\left(\log_3\left(x^2\right)\right) = \dfrac{2x}{x^2 \ln 3} = \dfrac{2}{x \ln 3}$

(3) $$\begin{aligned}\frac{d}{dx}\left(x \ln\left(x^2+1\right)\right) &= x \cdot \frac{d}{dx}\left(\ln\left(x^2+1\right)\right) + \ln\left(x^2+1\right) \cdot \frac{d}{dx}(x) \\ &= x \cdot \frac{2x}{x^2+1} + \ln\left(x^2+1\right) \cdot 1 \\ &= \frac{2x^2}{x^2+1} + \ln\left(x^2+1\right)\end{aligned}$$

Rules of Differentiation & Chain Rule

Example 15

(1) $\dfrac{d}{dx}\left(5^{2\log_5 x}\right) = \dfrac{d}{dx}\left(5^{\log_5\left(x^2\right)}\right) = \dfrac{d}{dx}\left(x^2\right) = 2x$

(2) $\dfrac{d}{dt}\left(\log_{10}\left(t\cdot 10^t\right)\right) = \dfrac{d}{dt}\left(\log_{10} t + \log_{10}\left(10^t\right)\right)$

$$= \frac{d}{dt}\left(\log_{10} t + t\right) = \frac{1}{t\cdot\ln 10} + 1$$

(3) $\dfrac{d}{dx}\left(\dfrac{x}{\ln x}\right) = \dfrac{(\ln x)\cdot 1 - x\cdot\frac{1}{x}}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$

Rules of Differentiation & Chain Rule

Rule [10]:

$$\frac{d}{dx}\Big(\sin g(x)\Big) = \quad g'(x)\cdot\cos(g(x))$$

$$\frac{d}{dx}\Big(\cos g(x)\Big) = -g'(x)\cdot\sin(g(x))$$

$$\frac{d}{dx}\Big(\tan g(x)\Big) = \quad g'(x)\cdot\sec^2(g(x))$$

$$\frac{d}{dx}\Big(\cot g(x)\Big) = -g'(x)\cdot\csc^2(g(x))$$

$$\frac{d}{dx}\Big(\sec g(x)\Big) = \quad g'(x)\cdot\sec(g(x))\tan(g(x))$$

$$\frac{d}{dx}\Big(\csc g(x)\Big) = -g'(x)\cdot\csc(g(x))\cot(g(x))$$

Rules of Differentiation & Chain Rule

Example 16

(1) $\dfrac{d}{dx}\Big(\tan x\Big) = \sec^2 x \cdot 1 = \sec^2 x$

(2)
$$\begin{aligned}\frac{d}{dx}\left(x\sin\left(2^x\right)\right) &= x \cdot \frac{d}{dx}\left(\sin\left(2^x\right)\right) + \sin\left(2^x\right) \cdot \frac{d}{dx}(x) \\ &= x \cdot \cos\left(2^x\right) \cdot 2^x \cdot \ln 2 + \sin\left(2^x\right) \cdot 1 \\ &= x\, 2^x \, \ln 2 \, \cos\left(2^x\right) + \sin\left(2^x\right)\end{aligned}$$

(3)
$$\begin{aligned}\frac{d}{dx}\left(\frac{1+\cos x}{\sin x}\right) &= \frac{d}{dx}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) \\ &= \frac{d}{dx}\Big(\csc x + \cot x\Big) \\ &= -\csc x \cot x - \csc^2 x\end{aligned}$$

Rules of Differentiation & Chain Rule

Example 17

(1) $$\frac{d}{dx}\left(4^{x\cos x}\right) = 4^{x\cos x} \cdot \ln 4 \cdot \frac{d}{dx}(x\cos x)$$
$$= 4^{x\cos x} \cdot \ln 4 \cdot \Big(-x \cdot \sin x + \cos x \Big)$$

(2) $$\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{(1+\cos x)\cdot(\sin x)' - (\sin x)(1+\cos x)'}{(1+\cos x)^2}$$
$$= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$
$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

Rules of Differentiation & Chain Rule

Example 18

(1) $$\frac{d}{dx}\left(\sin^3 x\right) = \frac{d}{dx}\left((\sin x)^3\right)$$
$$= 3(\sin x)^2 \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x$$

(2) $$\frac{d}{dx}\Big(\,(\sec x + \tan x)\,(\sec x - \tan x)\,\Big) = \frac{d}{dx}\left(\sec^2 x - \tan^2 x\right)$$
$$= \frac{d}{dx}(1) = 0$$

Rules of Differentiation & Chain Rule

Example 19

If $f(x) = \ln\left(\frac{e^x \sin x}{\sqrt{x+1}}\right)$. Find $f'(x)$.

$$\begin{aligned}
f(x) &= \ln\left(e^x \sin x\right) - \ln\left(\sqrt{x+1}\right) \\
&= \ln\left(e^x\right) + \ln\left(\sin x\right) - \ln\left((x+1)^{1/2}\right) \\
&= x + \ln\left(\sin x\right) - \frac{1}{2}\ln(x+1) \\
\therefore\ f'(x) &= \frac{d}{dx}\left(x + \ln\left(\sin x\right) - \frac{1}{2}\ln(x+1)\right) \\
&= 1 + \frac{\cos x}{\sin x} - \frac{1}{2}\,\frac{1}{x+1} = 1 + \cot x - \frac{1}{2(x+1)}
\end{aligned}$$

Rules of Differentiation & Chain Rule

Rule [11]: $\dfrac{d}{dx}\left(\tan^{-1} g(x)\right) = \dfrac{g'(x)}{1+g^2(x)}$

$$\frac{d}{dx}\left(\sin^{-1} g(x)\right) = \frac{g'(x)}{\sqrt{1-g^2(x)}}$$

$$\frac{d}{dx}\left(\sec^{-1} g(x)\right) = \frac{g'(x)}{|g(x)|\sqrt{g^2(x)-1}}$$

Example 20

$$\frac{d}{dx}\left(\sin^{-1}(x^3)\right) = \frac{(x^3)'}{\sqrt{1-(x^3)^2}} = \frac{3x^2}{\sqrt{1-x^6}}$$

Rules of Differentiation & Chain Rule

Example 21

$$\begin{aligned}
\frac{d}{dx}\left(x\tan^{-1}\left(\sqrt{x}\right)\right) &= x\cdot\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{x}\right)\right)+\left(\tan^{-1}\left(\sqrt{x}\right)\right)\cdot\frac{dx}{dx}(x)\\
&= x\cdot\frac{\left(\sqrt{x}\right)'}{1+\left(\sqrt{x}\right)^2}+\tan^{-1}\left(\sqrt{x}\right)\\
&= x\cdot\frac{\frac{1}{2\sqrt{x}}}{1+x}+\tan^{-1}\left(\sqrt{x}\right)\\
&= x\cdot\frac{1}{2\sqrt{x}}\cdot\frac{1}{1+x}+\tan^{-1}\left(\sqrt{x}\right)\\
&= \frac{\sqrt{x}}{2(1+x)}+\tan^{-1}\left(\sqrt{x}\right)
\end{aligned}$$

Rules of Differentiation & Chain Rule

NOTE: $\lim_{h\to 0} \frac{f(x+ah)-f(x)}{bh} = \frac{a}{b}\, f'(x)$

Example 22

Let $f(x) = \ln x$. Evaluate $\lim_{h\to 0} \frac{f(1-2h)-f(1)}{3h}$

$$\lim_{h\to 0} \frac{f(1-2h)-f(1)}{3h} = \frac{-2}{3} f'(1)$$

$$\text{But} \quad f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$$

$$\therefore \lim_{h\to 0} \frac{f(1-2h)-f(1)}{3h} = \frac{-2}{3} \cdot 1 = -\frac{2}{3}$$

Rules of Differentiation & Chain Rule

Example 23

Given that $g(2) = {}^{1}\!/_{2}$, $g'(2) = -{}^{1}\!/_{4}$, $f'\left({}^{1}\!/_{2}\right) = 1$, find $\Big(f \circ g\Big)'(2)$

$$\Big(f \circ g\Big)'(2) = f'(g(2)) \cdot g'(2) = f'\left({}^{1}\!/_{2}\right) \cdot {}^{-1}\!/_{4} = 1 \cdot {}^{-1}\!/_{4} = {}^{-1}\!/_{4}$$

Example 24

Let $\dfrac{d}{dx}\left(f\left(x^2\right)\right) = x^2$. Find $f'\left(x^2\right)$.

$$\frac{d}{dx}\left(f\left(x^2\right)\right) = x^2 \Rightarrow f'\left(x^2\right) \cdot \frac{d}{dx}\left(x^2\right) = x^2$$

$$\Rightarrow 2xf'\left(x^2\right) = x^2 \Rightarrow f'\left(x^2\right) = \frac{x}{2}$$

Rules of Differentiation & Chain Rule

Example 25

For what values of a and b, the function $f(x) = \begin{cases} x^2 + a & : x \leq 1 \\ bx & : x > 1 \end{cases}$ is differentiable at $x = 1$.

$f'(1)$ **exists.**

$f'_+(1) = f'_-(1)$

$$f'(x) = \begin{cases} 2x & : x < 1 \\ b & : x > 1 \\ ? & : x = 1 \end{cases}$$

$b = 2(1) = 2$

$f(x)$ **continuous at** $x = 1$

$\lim_{x \to 1} f(x)$ exists

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

$b = 1^2 + a$

$a = b - 1 = 2 - 1 = 1$

Rules of Differentiation & Chain Rule

Exercise 1

(1) *If* $f(x) = \sqrt{x}g(x)$ *where* $g(4) = 2$, $g'(4) = 4$, *find* $f'(4)$.

(2) *Find the equation of the tangent line of* $f(x) = \dfrac{x-1}{x-2}$ *at* $x = 3$.

(3) *Find the points on the curve* $y = 2x^3 + 3x^2 - 12x + 1$ *where the tangent is horizontal.*

(4) *If the tangent line to* $f(x)$ *at* $(4, 3)$ *passes through the point* $(0, 2)$. *Find* $f(4)$, $f'(4)$, *and the equation of the tangent line.*

(5) *Let* $y = f(x^2 + 1)$. *If* $f(2) = 3$ *and* $f'(2) = 5$, *find* $\dfrac{dy}{dx}$ *at* $x = 1$.

Rules of Differentiation & Chain Rule

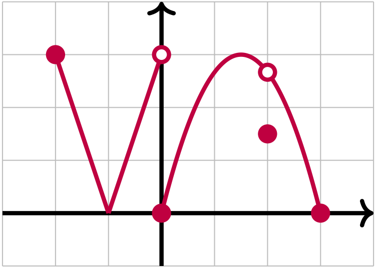
Exercise 2

The figure shows the graph of a function over a closed interval. At what domain points does the function appear to be

(1) *differentiable?*

(2) *continuous but not differentiable?*

(3) *neither continuous nor differentiable?*



Rules of Differentiation & Chain Rule

Exercise 3

Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

Find the derivatives with respect to x of the following combinations at the given value of x.

(1) $f(x) \cdot g^3(x)$ *at* $x = 0$

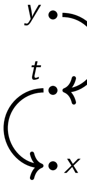
(2) $f(x + g(x))$ *at* $x = 0$

(3) $(x^{11} + f(x))^{-2}$ *at* $x = 1$

The Chain Rule (*Origin*)

If $y = f(t)$ and $t = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$



Example 26

If $y = \tan x$ and $x = 4t^3 + t$, find $\dfrac{dy}{dt}$.

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= \sec^2 x \cdot \left(12t^2 + 1\right) \\ &= \left(12t^2 + 1\right) \sec^2\left(4t^3 + t\right)\end{aligned}$$

The Chain Rule (*Origin*)

Example 27

Let $y = 5 - e^t$ and $t = 2x^2 - 3$. Find $\left.\dfrac{dy}{dx}\right|_{x=1}$.

Note that when $x = 1$, then $t = 2(1)^2 - 3 = -1$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = (-e^t) \cdot (4x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \left(-e^{-1}\right) \cdot (4) = \frac{-4}{e}$$

Exercise 4

If $\left.\dfrac{dy}{dx}\right|_{x=2} = 12$ *and* $t = x^2 + 1$. *Find* $\left.\dfrac{dy}{dt}\right|_{t=5}$.

Second Order Derivative

The **second order derivative** of $f(x)$ is denoted by

$$f''(x) \ , \ \ \frac{d^2}{dx^2}f(x) \ , \ \ f^{(2)}(x)$$

and is defined by $f''(c) = \lim_{h\to 0} \dfrac{f'(c+h) - f'(c)}{h}$

$$= \lim_{x\to c} \frac{f'(x) - f'(c)}{x - c}$$

and can be evaluated by the formula

$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big)$$

n^{th} Order Derivative

In general, the n^{th} order derivative of $f(x)$ is

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

Example 28

Let $f(x) = x^3 - 4x$. Find

(1) $f'(2)$

$$f'(x) = 3x^2 - 4$$

$$f'(2) = 3 \cdot (2)^2 - 4 = 8$$

(2) $f''(1)$

$$f''(x) = \frac{d}{dx}\left(3x^2 - 4\right) = 6x$$

$$f''(1) = 6 \times 1 = 6$$

(3) $f'''(x) = \frac{d}{dx}(6x) = 6$

(4) $f^{(4)}(x) = \frac{d}{dx}(6) = 0$

n^{th} Order Derivative

Example 29

Find $\dfrac{d^2}{dx^2}\left(\sin\left(x^2\right)\right)\Big|_{x=0}$

$$\frac{d}{dx}\left(\sin\left(x^2\right)\right) = \cos\left(x^2\right)\cdot\frac{d}{dx}\left(x^2\right) = 2x\cos\left(x^2\right)$$

$$\begin{aligned}\frac{d^2}{dx^2}\left(\sin\left(x^2\right)\right) &= \frac{d}{dx}\left(2x\cos\left(x^2\right)\right)\\ &= 2x\cdot\frac{d}{dx}\left(\cos\left(x^2\right)\right) + \cos\left(x^2\right)\cdot\frac{d}{dx}(2x)\\ &= -4x^2\sin\left(x^2\right) + 2\cos\left(x^2\right)\end{aligned}$$

$$\therefore \frac{d^2}{dx^2}\left(\sin\left(x^2\right)\right)\Big|_{x=0} = 0 + 2(1) = 2$$

n^{th} Order Derivative

Example 30

Find a formula for the n^{th} order derivative of $f(x) = xe^x$.

$$f'(x) = x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x) = xe^x + e^x = (x+1)e^x$$

$$\begin{aligned} f''(x) &= (x+1) \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x+1) = (x+1)e^x + e^x \\ &= (x+2)e^x \end{aligned}$$

$$\begin{aligned} f'''(x) &= (x+2) \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x+2) = (x+2)e^x + e^x \\ &= (x+3)e^x \end{aligned}$$

$$\therefore f^{(n)}(x) = (x+n)e^x$$

n^{th} Order Derivative

Example 31

Let $f(x) = \sin x$. Find $f^{(87)}(x)$.

$$f^{(0)}(x) = \sin x$$
$$f^{(1)}(x) = \cos x$$
$$f^{(2)}(x) = -\sin x$$
$$f^{(3)}(x) = -\cos x$$

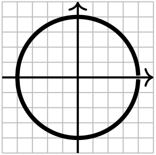
Since 87 mod 4 = 3, then
$f^{(87)}(x) = f^{(3)}(x) = -\cos x$.

NOTE: $$\frac{d^n}{dx^n}\big(\sin x\big) = \sin\left(\frac{n\pi}{2} + x\right)$$
$$\frac{d^n}{dx^n}\big(\cos x\big) = \cos\left(\frac{n\pi}{2} + x\right)$$

Implicit Equation

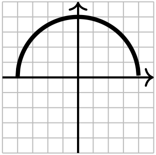
An equation in x and y can implicitly define more than one function of x, so it is not a graph of a function.

Example 32



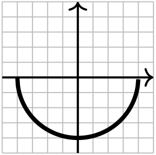
$x^2 + y^2 = 4$

$=$

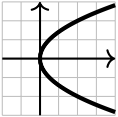


$y = \sqrt{4 - x^2}$

$\cup$

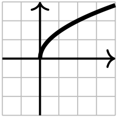


$y = -\sqrt{4 - x^2}$



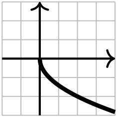
$x = y^2$

$=$



$y = \sqrt{x}$

$\cup$



$y = -\sqrt{x}$

The Derivative of Implicit Equation

(1) Differentiate both sides of the implicit equation, treating y as a function of x.

(2) Collect the terms with y'.

(3) Solve for y'.

Example 33

Find y' if $x = y^2$.

$$x = y^2 \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\left(y^2\right)$$
$$\Rightarrow 1 = 2y \cdot y'$$
$$\Rightarrow y' = \frac{1}{2y}$$

The Derivative of Implicit Equation

Example 34

Find $\dfrac{dy}{dx}$ if $y = \sin(xy)$.

$$
\begin{aligned}
\frac{d}{dx}(y) = \frac{d}{dx}(\sin(xy)) &\Rightarrow y' = \cos(xy) \cdot \frac{d}{dx}(xy) \\
&\Rightarrow y' = \cos(xy) \cdot (xy' + y) \\
&\Rightarrow y' = xy' \cos(xy) + y \cos(xy) \\
&\Rightarrow y' - xy' \cos(xy) = y \cos(xy) \\
&\Rightarrow y'(1 - x \cos(xy)) = y \cos(xy) \\
&\Rightarrow y' = \frac{y \cos(xy)}{1 - x \cos(xy)}
\end{aligned}
$$

The Derivative of Implicit Equation

Example 35

Find the equation of tangent line of $x^2 + y^2 = 25$ at $(3, 4)$.

* **Point:** $(3, 4)$
* **Slope:** $\dfrac{d}{dx}\left(x^2 + y^2\right) = \dfrac{d}{dx}(25) \Rightarrow 2x + 2yy' = 0$

$$\Rightarrow y' = \frac{-x}{y}$$

$$\therefore m = y'(3, 4) = {}^{-3}\!/_{4}$$

* **Equation:** $y - y_0 = m\,(x - x_0)$

$$y - 4 = {}^{-3}\!/_{4}(x - 3)$$

$$y = {}^{-3}\!/_{4}x + {}^{25}\!/_{4}$$

The Derivative of Implicit Equation

Example 36

Find $\dfrac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$.

$$\frac{d}{dx}\left(4x^2 - 2y^2\right) = \frac{d}{dx}(9)$$
$$8x - 4yy' = 0$$
$$y' = \frac{2x}{y}$$

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{2x}{y}\right)$$
$$y'' = \frac{y \cdot 2 - 2x \cdot y'}{y^2}$$
$$= \frac{2y - 2x \cdot \frac{2x}{y}}{y^2}$$
$$= \frac{2y^2 - 4x^2}{y^3} = \frac{-9}{y^3}$$

The Derivative of Implicit Equation

Rule: $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$

Example 37

Using the values in the table for a function $f(x)$, find $\left(f^{-1}\right)'(2)$.

x	$f(x)$	$f'(x)$
1	2	3
2	9	12

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(1)} = \frac{1}{3}$$

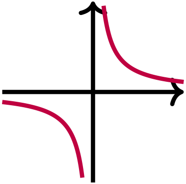
The Derivative of Implicit Equation

Example 38

Given that $y = x^{\sin x}$, find y'.

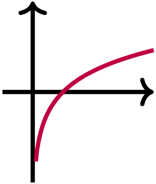
$$\begin{aligned} \ln y = \ln\left(x^{\sin x}\right) &\Rightarrow \ln y = \sin x \ln x \\ \frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \ln x) &\Rightarrow \frac{y'}{y} = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x \\ &\Rightarrow y' = y\left(\frac{\sin x}{x} + \cos x \ln x\right) \\ &\Rightarrow y' = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right) \end{aligned}$$

Indeterminate Forms & L'Hospital's Rule



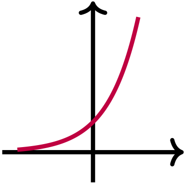
$$\frac{b}{0} = \infty$$

$$\frac{b}{\infty} = 0$$



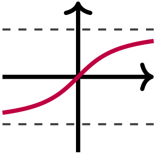
$$\ln(0) = -\infty$$

$$\ln(\infty) = \infty$$



$$e^{\infty} = \infty$$

$$e^{-\infty} = 0$$



$$\tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\tan^{-1}(-\infty) = -\frac{\pi}{2}$$

Indeterminate Forms & L'Hospital's Rule

Type [1]: $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty} \equiv \lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$

$$= \lim_{x\to c} \frac{f'(x)}{g'(x)} \textbf{ (by L'Hospital's Rule)}$$

Example 39

Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x^3} = \dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^x - 1\right)}{\frac{d}{dx}\left(x^3\right)} = \lim_{x\to 0} \frac{e^x}{3x^2} = \frac{1}{0}$$

$$= \infty \quad \textbf{d.n.e}$$

Indeterminate Forms & L'Hospital's Rule

Example 40

Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$

$$= \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

Example 41

Evaluate $\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty \quad \textbf{d.n.e}$$

Indeterminate Forms & L'Hospital's Rule

Type [2]: $0 \cdot \infty \equiv \lim_{x \to c} f(x) \cdot g(x) = \begin{cases} \lim_{x \to c} \dfrac{f(x)}{1/g(x)} = \dfrac{0}{0} \\ \lim_{x \to c} \dfrac{g(x)}{1/f(x)} = \dfrac{\infty}{\infty} \end{cases}$

Example 42

Evaluate $\lim_{x \to 0^+} x \ln x = 0 \cdot (-\infty)$

$$= \lim_{x \to 0^+} \frac{\ln x}{1/x} = \frac{-\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Indeterminate Forms & L'Hospital's Rule

Example 43

Evaluate $\lim_{x\to\infty} xe^{-x} = \infty \cdot 0$

$$= \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

Exercise 5

Evaluate (1) $\lim_{x\to 0} \dfrac{5x - \sin(5x)}{4x - \tan(4x)}$

(2) $\lim_{x\to \pi/4} (1 - tanx) \cdot \sec(2x)$

Indeterminate Forms & L'Hospital's Rule

Type [3]: $\begin{matrix} \infty - \infty \\ -\infty + \infty \end{matrix} \equiv \lim_{x \to c} \left(f(x) - g(x) \right)$ Common Denominator, Factoring, Conjugate

Example 44

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \infty - \infty$

$$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} = \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{x \cos x + \sin x} = \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{-\sin x}{-x \sin x + 2 \cos x} = \frac{0}{2} = 0$$

NOTE: $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$

Indeterminate Forms & L'Hospital's Rule

Example 45

Evaluate

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right) &= \infty-\infty \\
&= \lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)\cdot\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\
&= \lim_{x\to\infty}\frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\infty} = 0
\end{aligned}$$

Indeterminate Forms & L'Hospital's Rule

Example 46

Evaluate
$$\begin{aligned}\lim_{t\to\infty}\left(te^{1/t}-t\right) &= \infty-\infty \\ &= \lim_{t\to\infty} t\cdot\left(e^{1/t}-1\right) = \infty\cdot 0 \\ &= \lim_{t\to\infty}\frac{e^{1/t}-1}{1/t} = \frac{0}{0} \\ &= \lim_{t\to\infty}\frac{(-1/t^2)\cdot e^{1/t}}{-1/t^2} \\ &= \lim_{t\to\infty} e^{1/t} = e^0 = 1\end{aligned}$$

Indeterminate Forms & L'Hospital's Rule

Type [4]: $0^0\,,\ \infty^0\,,\ 1^\infty \equiv \lim_{x\to c}\left(f(x)^{g(x)}\right)$

Example 47

Evaluate $\lim_{x\to 0^+}\left(1+\sin x\right)^{1/x} = 1^\infty$

Let $\qquad y = \left(1+\sin x\right)^{1/x}$

$$\ln y = \ln\left((1+\sin x)^{1/x}\right) = \frac{\ln(1+\sin x)}{x}$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin x)}{x} = \lim_{x\to 0^+}\frac{\cos x}{1+\sin x} = 1$$

$$\lim_{x\to 0^+} y = e^1 = e$$

Indeterminate Forms & L'Hospital's Rule

Example 48

Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x} = 1^{\infty}$

Let $y = \left(1+\frac{2}{x}\right)^{3x} \Rightarrow \ln y = 3x\ln\left(1+\frac{2}{x}\right)$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}3x\ln\left(1+\frac{2}{x}\right) = \infty\cdot 0$$

$$= 3\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = 3\lim_{x\to\infty}\frac{\frac{-2/x^2}{1+2/x}}{-1/x^2} = 3\lim_{x\to\infty}\frac{2}{1+2/x} = 6$$

$$\lim_{x\to\infty}y = e^{6}$$

Indeterminate Forms & L'Hospital's Rule

NOTE: $$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$$

$$\lim_{x\to 0}\left(1+ax\right)^{b/x}=e^{ab}$$

Exercise 6

Evaluate (1) $\displaystyle\lim_{x\to 0^+}\left(e^x-2x\right)^{3/x}$

(2) $\displaystyle\lim_{x\to 0^+}\left(1+3\sin x\right)^{2\cot x}$

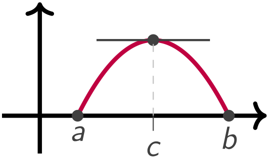
Rolle's Theorem

If $f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$f(a) = f(b) = 0$

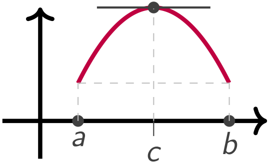
Then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.



NOTE:

(1) Rolle's Theorem still hold if $f(a) = f(b) \neq 0$.

(2) The derivative at any endpoint of a closed interval does not exist.



Rolle's Theorem

Example 49

Verify that the hypotheses of Rolle's Theorem are satisfied for the function $f(x) = x^2 - 8x + 15$ on the interval $[3, 5]$, and find all values of c in that interval that satisfy the conclusion of the theorem.

(1) Since $f(x)$ is a polynomial, then it is continuous on $[3, 5]$.

(2) $f'(x) = 2x - 8$ always exists $\forall x \in (3, 5)$ since it is a polynomial.

(3) $f(3) = f(5) = 0$.

$\therefore$ There exists $c \in (3, 5)$ such that

$$\begin{aligned} f'(c) &= 0 \\ 2c - 8 &= 0 \\ c &= 4 \in (3, 5) \end{aligned}$$

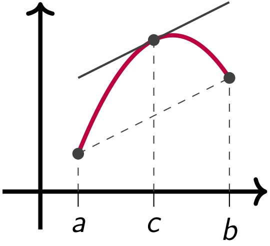
Mean Value Theorem (MVT)

If $f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

Then there exists at least one $c \in (a, b)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Example 50

Verify that the hypotheses of the MVT are satisfied for the function $f(x) = x^2 - x$ on the interval $[-3, 5]$, and find all values of c in the interval that satisfy the conclusion of the theorem.

$$f'(c) = \frac{f(5) - f(-3)}{5 - (-3)}$$

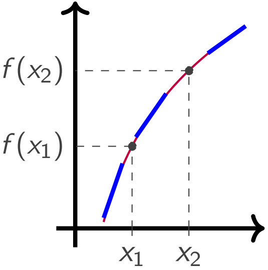
$$2c - 1 = \frac{20 - 12}{8}$$

$$2c - 1 = 1$$

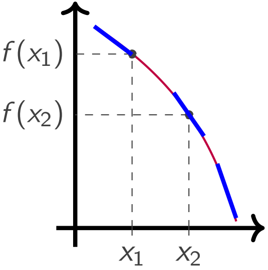
$$\therefore c = 1 \in (-3, 5)$$

Increasing and Decreasing Functions

* $f(x)$ **increases**
* If $x_1 < x_2$ then $f(x_1) < f(x_2)$
* All tangents have positive slopes
* $f'(x) > 0$



* $f(x)$ **decreases**
* If $x_1 < x_2$ then $f(x_1) > f(x_2)$
* All tangents have negative slopes
* $f'(x) < 0$



Increasing and Decreasing Functions

Theorem 2

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). If

(1) $f'(x) > 0$ *for all* $x \in (a, b)$, *then* f *is* ***increasing*** *on* $[a, b]$.

(2) $f'(x) < 0$ *for all* $x \in (a, b)$, *then* f *is* ***decreasing*** *on* $[a, b]$.

(3) $f'(x) = 0$ *for all* $x \in (a, b)$, *then* f *is* ***constant*** *on* $[a, b]$.

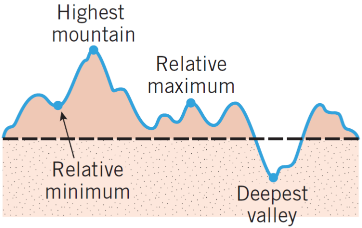
Definition 3

A **critical number** of a function f is any number c in the domain of f at which $\underbrace{f'(c) = 0}_{\text{Stationary}}$ or $f'(c)$ d.n.e

Maximum & Minimum Values

(1) A function f has **relative (local) maximum** at c if $f(c) \geq f(x)$ for all x in some open interval containing c.

(2) A function f has **relative (local) minimum** at c if $f(c) \leq f(x)$ for all x in some open interval containing c.



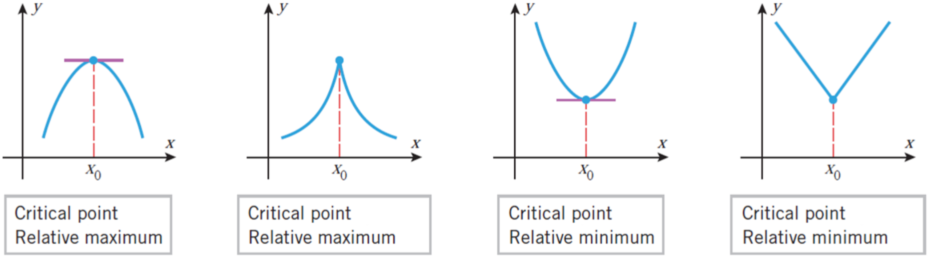
(3) A function f has **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f.

(4) A function f has **absolute minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f.

Maximum & Minimum Values

Theorem 3

*If f has a relative **extremum** (min or max) at x_0, then either $f'(x_0) = 0$ or $f'(x_0)$ d.n.e*



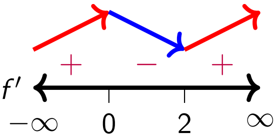
NOTE: A function f has a relative extremum at those critical points where f' changes sign.

Maximum & Minimum Values

Example 51

Determine the intervals where the function $f(x) = x^3 - 3x^2 + 2$ increasing and where it is decreasing, and find the extremum points.

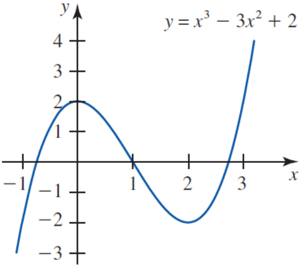
* The domain of f is $\mathbb{R}$.
* **Critical numbers:**
  $f'(x) = 3x^2 - 6x = 3x(x-2)$
  $f'(x) = 0$ if $x = 0, x = 2$
* $f'(x)$ always exists.



* f decreases on $[0, 2]$
* f increases on $(-\infty, 0], [2, \infty)$
* f has local max at $x = 0$ with $f(0) = 2$
* f has local min at $x = 2$ with $f(2) = 0$

Absolute Extremum

NOTE: The figure shows the graph of $f(x) = x^3 - 3x^2 + 2$ in the previous example.



Theorem 4

If f is continuous on a closed interval [a,b], then f attains an absolute maximum value $f(c)$ for some number $c \in [a, b]$ and an absolute minimum value $f(d)$ for some number $d \in [a, b]$.

Absolute Extremum

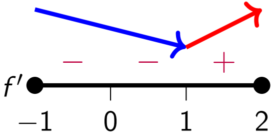
Example 52

Determine the intervals where the function $f(x) = 3x^4 - 4x^3 - 8$ increasing and where it is decreasing on $[-1, 2]$. Also, find the extremum points and identify their types.

Critical numbers:

$$f'(x) = 12x^3 - 12x^2$$
$$= 12x^2(x-1)$$

$f'(x) = 0$ if $x = 0, x = 1$



* f increases on $[1, 2]$

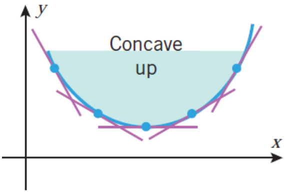
* f decreases on $[-1, 1]$

*

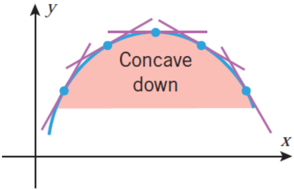
x	$f(x)$	Type
-1	-1	local max
2	8	abs. max
1	-9	abs. min

Concavity: Up & Down

* Increasing Slopes
* $(f')' > 0 \Rightarrow f''(x) > 0$



* decreasing Slopes
* $(f')' < 0 \Rightarrow f''(x) < 0$



Concavity: Up & Down

Theorem 5

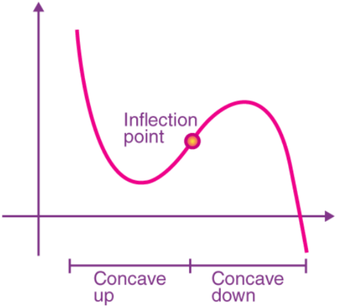
Let f be a function whose 2^{nd} derivative exists on (a, b).

(1) *If $f''(x) > 0$ for all $x \in (a, b)$ then the graph of f is concave upward on (a, b).*

(2) *If $f''(x) < 0$ for all $x \in (a, b)$ then the graph of f is concave downward on (a, b).*

Definition 4

Inflection point is a point in which the concavity of the function changes.



Concavity: Up & Down

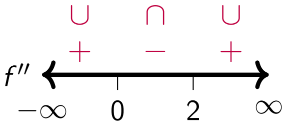
Theorem 6

If $(c, f(c))$ is a point of inflection of the graph of f then either $f''(c) = 0$ or $f''(c)$ does not exist.

Example 53

Determine the inflection points and the intervals of concavity for $f(x) = x^4 - 4x^3 + 12$.

$$
\begin{aligned}
f'(x) &= 4x^3 - 12x^2 \\
f''(x) &= 12x^2 - 24x \\
&= 12x(x-2) \\
f''(x) &= 0 \text{ if } x = 0, x = 2
\end{aligned}
$$



* f concave-up on $(-\infty, 0], [2, \infty)$
* f concave-down on $[0, 2]$
* $(0, 12), (2, -4)$ are inflection points

Example to Find them All

Example 54

Let $f(x) = 2x^3 - 6x$; $x \in [-3, 4]$. Find the:

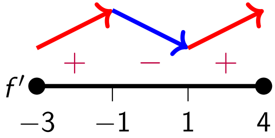
(1) critical numbers

(2) intervals of increasing and decreasing

(3) maximum & minimum values and classify them

(4) intervals of concavity and inflection points

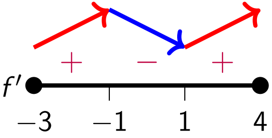
$$f'(x) = 0$$
$$6x^2 - 6 = 0$$
$$x = \pm 1 \text{ (**Critical Numbers**)}$$



Example to Find them All

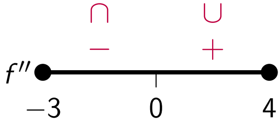
* f increases on $[-3,-1],[1,4]$
* f decreases on $[-1,1]$
*

x	$f(x)$	Type
-3	-36	min (abs)
1	-4	min (local)
-1	4	max (local)
4	104	max (abs)



Example to Find them All

$$f(x) = 2x^3 - 6x$$
$$f'(x) = 6x^2 - 6$$
$$f''(x) = 12x = 0$$
$$x = 0$$



* f concave-up on $[0, 4]$
* f concave-down on $[-3, 0]$
* $(0, 0)$ is inflection point