Course: Calculus (3)

Chapter: [11] THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.1] RECTANGULAR COORDINATES IN 3-SPACE; SPHERES; CYLINDRICAL SURFACES

RECTANGULAR COORDINATE SYSTEMS

In the remainder of this slides, we will call:

- three-dimensional space: 3-space
- two-dimensional space (a plane): **2-space**
- one-dimensional space (a line): **1-space**

Points in 3-space can be placed in one-toone correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the x —axis, the y —axis, and the z —axis, positioned so that their origins coincide.



RECTANGULAR COORDINATE SYSTEMS



- The three coordinate axes form a threedimensional rectangular coordinate system (or Cartesian coordinate system).
- The point of intersection of the coordinate axes is called the origin of the coordinate system.

RECTANGULAR COORDINATE SYSTEMS



| xy-planeConsists of all points of the form $(x, y, 0)$ xz-planeConsists of all points of the form $(x, 0, z)$ yz-planeConsists of all points of the form $(0, y, z)$ x-axisConsists of all points of the form $(x, 0, 0)$ y-axisConsists of all points of the form $(0, y, 0)$ z-axisConsists of all points of the form $(0, y, 0)$ z-axisConsists of all points of the form $(0, y, 0)$ | REGION | DESCRIPTION |
|--|--|--|
| | <i>xy</i> -plane <i>xz</i> -plane <i>yz</i> -plane <i>x</i> -axis <i>y</i> -axis <i>z</i> -axis | Consists of all points of the form $(x, y, 0)$ Consists of all points of the form $(x, 0, z)$ Consists of all points of the form $(0, y, z)$ Consists of all points of the form $(x, 0, 0)$ Consists of all points of the form $(0, y, 0)$ Consists of all points of the form $(0, 0, z)$ |



 $P_1(x_1, y_1, z_1)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example Find the distance *d* between the points (2, 3, -1) and (4, -1, 3).

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + (3-(-1))^2}$$
$$= \sqrt{4+16+16}$$
$$= 6$$



Midpoint
$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example

 $x_1 y_1 z_1$ $x_2 y_2 z_2$ Find the midpoint between the points (2, 3, -1) and (4, -1, 3).

$$midpoint = \left(\frac{2+4}{2}, \frac{3+(-1)}{2}, \frac{-1+3}{2}\right)$$
$$= (3,1,1)$$



$$(x-a)^2 + (y-b)^2 = r^2$$

Circle in 2-space



$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Sphere in 3-space



$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Sphere in 3-space

Example

Find the equation of the sphere with center (1, -2, -4) and radius 3.

$$(x-1)^{2} + (y+2)^{2} + (z+4)^{2} = 9$$
$$x^{2} + y^{2} + z^{2} - 2x + 4y + 8z = -12$$

Example

Find the center and radius of the sphere $(x-5)^2 + y^2 + (z+3)^2 = 5$

Center(, ,)Radius $\sqrt{5}$

It is possible to graph equations in two variables in 3 — space.

Example: $x^2 + y^2 = 1$

Observe that the equation does not impose any restrictions on *z*.

This means that we can obtain the graph of $x^2 + y^2 = 1$ in an xyz –coordinate system by first graphing the equation in the xy –plane.



It is possible to graph equations in two variables in 3 — space.

Example: $x^2 + y^2 = 1$

Observe that the equation does not impose any restrictions on *z*.

This means that we can obtain the graph of $x^2 + y^2 = 1$ in an xyz –coordinate system. by first graphing the equation in the xy –plane.

And then translating that graph parallel to the z —axis to generate the entire graph.



Example $x^2 + z^2 = 1$



Example $z = y^2$



Example $z = \sin x$



11.1.2 THEOREM An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.

Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.2] VECTORS

A **Vector** in 2-space or 3-space; is an **arrow** with direction and length (magnitude).



Two vectors **v** and **w** are equal if they have the same length and same direction, and we write $\mathbf{v} = \mathbf{w}$.



Two vectors are equal if they are translations of one another.

Because vectors are **not** affected by translation, the initial point of a vector \mathbf{v} can be *moved* to any convenient point A by making an appropriate *translation*.





NOTE:

• If the *initial* and *terminal* points of a vector coincide, then the

vector has **length zero**; we call this the **zero vector** and denote it by

0.

• The zero vector does not have a specific direction

11.2.2 DEFINITION If **v** is a nonzero vector and *k* is a nonzero real number (a scalar), then the *scalar multiple kv* is defined to be the vector whose length is |k| times the length of **v** and whose direction is the same as that of **v** if k > 0 and opposite to that of **v** if k < 0. We define $k\mathbf{v} = \mathbf{0}$ if k = 0 or $\mathbf{v} = \mathbf{0}$.



NOTE: The vectors **v** and *k***v** are parallel vectors.

VECTORS IN COORDINATE SYSTEMS

If a vector **v** is positioned with its initial point at the *origin* of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2, v_3) .



We call these coordinates the *components of* **v**, and we write **v** in component form using the *bracket* notation

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

VECTORS IN COORDINATE SYSTEMS

NOTE: 0 = (0, 0, 0)

11.2.3 THEOREM *Two vectors are equivalent if and only if their corresponding components are equal.*

Example: Find the values of
$$a, b, c$$
 if $\langle -2, b, 3 \rangle = \langle a, 0, c \rangle$.

ARITHMETIC OPERATIONS ON VECTORS

11.2.4 THEOREM If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are vectors in 2-space and k is any scalar, then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$
(1)

$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle$$
(2)

$$k\mathbf{v} = \langle kv_1, kv_2 \rangle$$
(3)

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space and k is any scalar, then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$
(4)

$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$$
(5)

$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$$
(6)

ARITHMETIC OPERATIONS ON VECTORS

Example: If $\mathbf{v} = \langle 2, 0, 1 \rangle$ and $\mathbf{w} = \langle 3, 5, -4 \rangle$, then

1.
$$\mathbf{v} + \mathbf{w} = \langle 2, \mathbf{0}, \mathbf{1} \rangle + \langle 3, \mathbf{5}, -\mathbf{4} \rangle = \langle 5, 5, -3 \rangle$$

2.
$$\mathbf{v} - 2\mathbf{w} = \langle 2, \mathbf{0}, \mathbf{1} \rangle - 2\langle 3, \mathbf{5}, -\mathbf{4} \rangle$$

= $\langle 2, \mathbf{0}, \mathbf{1} \rangle - \langle 6, \mathbf{10}, -\mathbf{8} \rangle$
= $\langle -4, -10, 9 \rangle$

VECTORS WITH INITIAL POINT NOT AT THE ORIGIN

Example:

The vector from the point A(0, -2, 5) to the point B(3, 4, -1) is

$$\overrightarrow{AB} = \langle 3 - 0, 4 - (-2), -1 - 5 \rangle$$
$$= \langle 3, 6, -6 \rangle$$

$$P_{2}(x_{2}, y_{2})$$

$$P_{1}(x_{1}, y_{1})$$

$$P_{1}P_{2}$$

$$(x_{2} - x_{1}, y_{2} - y_{1})$$

$$\overrightarrow{P_{1}P_{2}}$$

$$Q$$

$$\overrightarrow{P_{1}P_{2}} = \langle x_{2} - x_{1}, y_{2} - y_{1} \rangle$$

RULES OF VECTOR ARITHMETIC

11.2.6 THEOREM For any vectors **u**, **v**, and **w** and any scalars k and l, the following relationships hold:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (e) $k(l\mathbf{u}) = (kl)\mathbf{u}$ (f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ (h) $1\mathbf{u} = \mathbf{u}$

NORM OF A VECTOR

The distance between the initial and terminal points of a vector v is called the length, the norm, or the magnitude of v and is denoted by ||v||.



$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

NORM OF A VECTOR

NOTE $||k\mathbf{v}|| = |k|||\mathbf{v}||$

Example: If $w = \langle 2,3,6 \rangle$ then find the norm of

1 W
$$||w|| = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{49} = 7$$

2
$$-3\mathbf{w}$$
 $||-3\mathbf{w}|| = |-3| \times ||\mathbf{w}|| = 3 \times 7 = 21$

UNIT VECTORS

- A vector of length 1 is called a unit vector.
- In an xy –coordinate system the unit vectors along the x and y –axes are denoted by i and j, respectively.

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \mathbf{j} \qquad \mathbf{j$$

 In an xyz – coordinate system the unit vectors along the x –, y – and z –axes are denoted by i, j and k, respectively.

$$\mathbf{k} = \langle 1, 0, 0 \rangle$$

$$\mathbf{k} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

UNIT VECTORS

NOTE Every vector in 2 —space is expressible uniquely in terms of **i** and **j** as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle$$
$$= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j}$$

Also, every vector in 3 —space is expressible uniquely in terms of \mathbf{i} , \mathbf{j} and \mathbf{k} as follows:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

UNIT VECTORS

Example:

 $\langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$ $\langle 2, -3, 4 \rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ $\langle -4, 0 \rangle = -4i + 0i = -4i$ $\langle 0, 3, 0 \rangle = 3\mathbf{i}$ (0, 0, 0) = 0i + 0j + 0k = 05(6i - 2j) = 30i - 10j(3i + 2j - k) - (4i - j + 2k) = -i + 3j - 3k $\|\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

NORMALIZING A VECTOR

The *unit vector* **u** that has the *same direction* as some given nonzero vector **v** is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

The process of obtaining a unit vector with the same direction of \mathbf{v} is called *normalizing* \mathbf{v} .

Example: Find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$
 $\therefore \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

VECTORS DETERMINED BY LENGTH AND ANGLE

$$\cos \theta = \frac{x}{\|\mathbf{v}\|} \Rightarrow x = \|\mathbf{v}\| \cos \theta$$
$$\sin \theta = \frac{y}{\|\mathbf{v}\|} \Rightarrow y = \|\mathbf{v}\| \sin \theta$$

 $\therefore \mathbf{v} = \langle \|\mathbf{v}\| \cos \theta , \|\mathbf{v}\| \sin \theta \rangle$



VECTORS DETERMINED BY LENGTH AND ANGLE

 $\therefore \mathbf{v} = \langle \|\mathbf{v}\| \cos \theta , \|\mathbf{v}\| \sin \theta \rangle$

Example:

Find the vector of length 2 that makes an angle of $\frac{\pi}{4}$ with the positive x —axis.

$$\mathbf{v} = \left\langle 2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4} \right\rangle = \left\langle \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right\rangle$$
$$= \left\langle \sqrt{2}, \sqrt{2} \right\rangle$$



Course: Calculus (3)

Chapter: [11] THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.3] DOT PRODUCT; PROJECTIONS
DEFINITION OF THE DOT PRODUCT

In this section we will define a *new kind of multiplication* in which two vectors are multiplied to produce a scalar.

11.3.1 DEFINITION If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the *dot product* of \mathbf{u} and \mathbf{v} is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

 $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then their dot product is defined as $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Example: $(3, 5) \cdot (-1, 2) = 3(-1) + 5(2) = 7$

$$\langle 2, 3 \rangle \cdot \langle -3, 2 \rangle = 2(-3) + 3(2) = 0$$

(i - 3j + 4k) \cdot (i + 5j + 2k) = 1(1) + (-3)(5) + 4(2) = -6

11.3.2 THEOREM If **u**, **v**, and **w** are vectors in 2- or 3-space and k is a scalar, then:

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b)
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

(c)
$$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$$

(d)
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$
 $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

(e) $\mathbf{0} \cdot \mathbf{v} = 0$

Example: Given that $\|\mathbf{a}\| = 5$, $\|\mathbf{b}\| = 10$ and $\mathbf{a} \cdot \mathbf{b} = -48$. Find

$$(3a + b) \cdot (a - 2b) = 3a \cdot a - 3a \cdot 2b + b \cdot a - 2b \cdot b$$

= 3||a||² - 6(a \cdot b) + (a \cdot b) - 2||b||²
= (3)(25) - (5)(-48) - (2)(100)
= 115

- Suppose that **u** and **v** are nonzero vectors in 2 space or 3 space that *are positioned so their initial points coincide*.
- We define the angle between **u** and **v** to be the angle θ determined by the vectors that satisfies the condition $\theta \in [0, \pi]$.



11.3.3 THEOREM If **u** and **v** are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and

(a)
$$\mathbf{v} = -\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$$

 $\mathbf{u} \cdot \mathbf{v} = (1)(-1) + (-2)(-5) + (3)(4) = 21$
 $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$
 $\|\mathbf{v}\| = \sqrt{(-1)^2 + (-5)^2 + 4^2} = \sqrt{42}$
 $\hat{\mathbf{u}} = \frac{\pi}{6}$
 $\hat{\mathbf{u}} = \frac{\pi}{6}$

11.3.3 THEOREM If **u** and **v** are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and

(b)
$$\mathbf{w} = 2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$$

 $\mathbf{u} \cdot \mathbf{w} = (1)(2) + (-2)(7) + (3)(4) = 0$ $\therefore \cos \theta = 0$
 $\theta = \frac{\pi}{2}$

11.3.3 THEOREM If **u** and **v** are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and

(c)
$$\mathbf{v} = 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

 $\mathbf{u} \cdot \mathbf{v} = (1)(4) + (-2)(6) + (3)(-2) = -14$
 $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$
 $\|\mathbf{v}\| = \sqrt{4^2 + 6^2 + (-2)^2} = \sqrt{56}$
 $\therefore \cos \theta = \frac{-14}{\sqrt{14} \times \sqrt{56}} = -\frac{1}{2}$
 $\theta = \frac{2\pi}{3}$

ANGLE BETWEEN VECTORS $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$



DIRECTION ANGLES



DIRECTION ANGLES

NOTE:

 $\cos^2 \alpha + \cos^2 \beta$

11.3.4 THEOREM The direction cosines of a nonzero vector $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ are

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos \beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos \gamma = \frac{v_3}{\|\mathbf{v}\|}$$
$$+ \cos^2 \gamma = 1$$

DIRECTION ANGLES

Example: Find the direction cosines of the vector $\mathbf{v} = \sqrt{3} \mathbf{i} + \mathbf{k}$.

$$\|\mathbf{v}\| = \sqrt{3+1} = 2$$
 $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$

$$\cos \alpha = \frac{\sqrt{3}}{2} \longrightarrow \alpha = \frac{\pi}{6} = 30^{\circ}$$
$$\cos \beta = 0 \longrightarrow \beta = \frac{\pi}{2} = 90^{\circ}$$
$$\cos \gamma = \frac{1}{2} \longrightarrow \gamma = \frac{\pi}{3} = 60^{\circ}$$

The angle between \mathbf{v} and x —axis

The angle between \mathbf{v} and y —axis

The angle between \mathbf{v} and z —axis

ORTHOGONAL PROJECTIONS



proj_ba || b
proj_ba = kb
proj_ba =
$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}$$
 b

The orthogonal projection of **a** on an arbitrary nonzero vector **b**.





ORTHOGONAL PROJECTIONS



$$\mathbf{w} \perp \mathbf{b}$$

proj $\mathbf{w}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{h}}\mathbf{a}$

The vector component of **a** orthogonal to **b**.

ORTHOGONAL PROJECTIONS

Example: Find the orthogonal projection of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, and then find the vector component of \mathbf{v} orthogonal to \mathbf{b} .

$$\mathbf{v} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = 2 + 2 + 0 = 4$$

 $\|\mathbf{b}\|^2 = 2^2 + 2^2 = 8$

Thus, the orthogonal projection of **v** on **b** is

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + \mathbf{j}$$

and the vector component of **v** orthogonal to **b** is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j}) = \mathbf{k}$$

Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.4] CROSS PRODUCT

DETERMINANTS

 A matrix is a rectangular array (table) of numbers arranged in rows and columns.

• For example,
$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & 5 & -7 \end{bmatrix}$$
.

• The **determinant** is a function that assigns numerical value to square matrix (number of rows = number of columns) of numbers.

• We define a
$$2 \times 2$$
 determinant by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

• For example,
$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (-2)(4) = 15 + 8 = 23$$

DETERMINANTS

A 3×3 determinant is defined in terms of 2×2 determinants by

$$\begin{vmatrix} a_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$



$$= 3 (8 - (-12)) + 2 (2 - 0) - 5 (3 - 0)$$

= 49

DETERMINANTS

11.4.1 THEOREM

- (a) If two rows in the array of a determinant are the same, then the value of the determinant is 0.
- (b) Interchanging two rows in the array of a determinant multiplies its value by -1.

CROSS PRODUCT

11.4.2 DEFINITION If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example Let $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find $\mathbf{u} \times \mathbf{v}$ $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$ $= 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$

- Keep in mind the essential differences between the *cross product* and the *dot product*:
 - ✓ The cross product is defined only for vectors in 3 —space, whereas the dot product is defined for vectors in 2 —space and 3 —space.
 - The cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar.

11.4.3 THEOREM If **u**, **v**, and **w** are any vectors in 3-space and k is any scalar, then:

- (a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- (c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- (d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
- (e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
- $(f) \quad \mathbf{u} \times \mathbf{u} = \mathbf{0}$

Example: Given that $\mathbf{a} \times \mathbf{b} = \langle -1, 2, 1 \rangle$. Find $(2\mathbf{a} - 3\mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})$.

$$(2a - 3b) \times (a + 2b) = 2a \times a + 2a \times 2b - 3b \times a - 3b \times 2b$$

= 2(a × a) + 4(a × b) -3(b × a) - 6(b × b)
= (2)(0) + (4)(a × b) + (3)(a × b) - (6)(0)
= (7)(a × b)

 $=\langle -7,14,7\rangle$

The following cross products occur so frequently that it is helpful to be familiar with them:

$$i \times j = k$$
 $j \times k = i$ $k \times i = j$ $j \times i = -k$ $k \times j = -i$ $i \times k = -j$



WARNING

- We can write a product of three real numbers as abc since the associative law (ab)c = a(bc) ensures that the same value for the product results no matter how the factors are grouped.
- The *associative* law does not hold for cross products. For example,

 $i \times (j \times j) = i \times 0 = 0$ $(i \times j) \times j = k \times j = -i$

• Thus, we cannot write a cross product with three vectors as $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$, since this expression is ambiguous (مُبهم) without parentheses.

11.4.4 THEOREM If **u** and **v** are vectors in 3-space, then: (a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to \mathbf{u})

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to \mathbf{v})





Example Find a vector that is orthogonal to both of the vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -7, 2, -1 \rangle$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix} \mathbf{k} = -5\mathbf{i} - 19\mathbf{j} - 3\mathbf{k}$$

11.4.5 THEOREM Let **u** and **v** be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

(a) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$

11.4.5 THEOREM Let **u** and **v** be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

- (a) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$
- (b) The area A of the parallelogram that has **u** and **v** as adjacent sides is

$$A = \|\mathbf{u} \times \mathbf{v}\|$$
$$\mathbf{v} = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|$$

(c) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if \mathbf{u} and \mathbf{v} are parallel vectors, that is, if and only if they are scalar multiples of one another.

Example Find the area of the triangle that is determined by the points $P_1(2,2,0), P_2(-1,0,2), \text{ and } P_3(0,4,3).$

$$\overrightarrow{P_1P_2} = \langle -3, -2, 2 \rangle$$

$$\overrightarrow{P_1P_3} = \langle -2, 2, 3 \rangle$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -10, 5, -10 \rangle$$

$$A = \frac{1}{2} \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \| = \frac{15}{2}$$



Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.*] REVIEW OF PARAMETRIC EQUATIONS

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations

x = f(t) and y = g(t)

are **parametric equations** and *t* is the **parameter.** The set of points (x, y) obtained as *t* varies over the interval *I* is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve**, denoted by *C*.

PARAMETRIC EQUATIONS

Example Express the graph of $y = x^2$ where $x \ge 0$ as parametric equations.

Let
$$x = t$$
 $y = t^2$ $t \ge 0$



PARAMETRIC EQUATIONS

Example The *counter-clockwise* orientation parametric equations of the



Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.5] PARAMETRIC EQUATIONS OF LINES

LINES DETERMINED BY A POINT AND A VECTOR

The parametric equations of the line in 2 -space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ are

$$x = x_0 + at$$
 , $y = y_0 + bt$



LINES DETERMINED BY A POINT AND A VECTOR

The parametric equations of the line in 3 - space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are

$$x = x_0 + at$$
 , $y = y_0 + bt$, $z = z_0 + ct$

Example Find parametric equations of the line passing through (1, 2, -3) and parallel to $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$.

$$x = 1 + 4t$$
 , $y = 2 + 5t$, $z = -3 - 7t$
Example

1. Find parametric equations of the line ℓ passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

The vector
$$\overrightarrow{P_1P_2} = \langle 5 - 2, 0 - 4, 7 - (-1) \rangle = \langle 3, -4, 8 \rangle$$
 is parallel to ℓ .

If
$$P_1$$
 is chosen:
 $x = 2 + 3t_1$
 $y = 4 - 4t_1$
 $z = -1 + 8t_1$

If P_2 is chosen:
 $4 - 4t_1 = -4t_2$
 $-1 + t_1 = t_2$

 $x = 5 + 3t_2$
 $y = -4t_2$
 $z = 7 + 8t_2$

Example

- 1. Find parametric equations of the line ℓ passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.
- 2. Where does the line intersect the xy plane?

$$z = 0$$
 $7 + 8t_2 = 0$ $t_2 = \frac{-7}{8}$

The point is $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$

$$x = 5 + 3t_2$$

 $y = -4t_2$
 $z = 7 + 8t_2$

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1: x = 1 + 4t, y = 5 - 4t, z = -1 + 5t$$
 $\mathbf{v}_1 = \langle 4, -4, 5 \rangle$
 $\ell_2: x = 2 + 8t, y = 4 - 3t, z = 5 + t$ $\mathbf{v}_2 = \langle 8, -3, 1 \rangle$

1. Are the lines parallel?

$$\ell_1 \parallel \ell_2 \iff \mathbf{v}_1 \parallel \mathbf{v}_2 \iff \mathbf{v}_2 = c \, \mathbf{v}_1$$

4c = 8 -4c = -35c = 1No such c $\therefore \ell_1 \text{ and } \ell_2 \text{ are NOT parallel lines.}$

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1$$
: $x = 1 + 4t$, $y = 5 - 4t$, $z = -1 + 5t$
 ℓ_2 : $x = 2 + 8t$, $y = 4 - 3t$, $z = 5 + t$

2. Do the lines intersect?

Suppose the point of intersection is

$$1 + 4t_1 = x^* = 2 + 8t_2$$

$$5 - 4t_1 = y^* = 4 - 3t_2$$

$$-1 + 5t_1 = z^* = 5 + t_2$$

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1$$
: $x = 1 + 4t$, $y = 5 - 4t$, $z = -1 + 5t$
 ℓ_2 : $x = 2 + 8t$, $y = 4 - 3t$, $z = 5 + t$

2. Do the lines intersect?

Suppose the point of intersection is

$$\begin{bmatrix} 1+4t_1 = 2+8t_2 \\ 5-4t_1 = 4-3t_2 \end{bmatrix} \xleftarrow{6 = 6+5t_2} \quad \textbf{BUT !!} \\ t_2 = 0 \\ t_1 = \frac{1}{4} \end{bmatrix} \text{ Do not satisfy the 3^{rd} equation.} \\ t_1 = \frac{1}{4} \end{bmatrix} \therefore \ell_1 \text{ and } \ell_2 \text{ do NOT intersect.}$$

- Two lines in 3 —space that are not parallel and do not intersect are called skew lines.
- Any two skew lines lie in *parallel planes*.



Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.6] PLANES IN 3-SPACE

PLANES PARALLEL TO THE COORDINATE PLANES



- A plane in 3 —space can be determined uniquely by specifying a *point* in the plane and a *vector perpendicular* to the
 - plane.
- A vector perpendicular to a plane is called a normal to the plane.



$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the point-normal form of the equation of a plane.



Example Find an equation of the plane passing through the point (3, -1, 7) and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5 \rangle$.

$$4(x-3) + 2(y+1) - 5(z-7) = 0$$

$$4x - 12 + 2y + 2 - 5z + 35 = 0$$

$$4x + 2y - 5z + 25 = 0$$

Example Determine whether the two planes are parallel.

$$P_1: 3x - 4y + 5z = 0$$
 $\mathbf{n}_1 = \langle 3, -4, 5 \rangle$ $P_2: -6x + 8y - 10z - 4 = 0$ $\mathbf{n}_2 = \langle -6, 8, -10 \rangle$

$$P_1 \parallel P_2 \quad \Leftrightarrow \quad \mathbf{n}_1 \parallel \mathbf{n}_2 \quad \Leftrightarrow \quad \mathbf{n}_2 = k \mathbf{n}_1$$

$$\Leftrightarrow \quad \langle -6, 8, -10 \rangle = k \langle 3, -4, 5 \rangle$$



 \Leftrightarrow k = -2 \therefore P_1 and P_2 are parallel planes

Example Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

$$\mathbf{n} = \overrightarrow{P_2 P_1} \times \overrightarrow{P_2 P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -2 \\ 1 & -4 & 1 \end{vmatrix} = \langle -9, -1, 5 \rangle$$

By using this normal and the point $P_3(3, -1, 2)$ in the plane, we obtain the point-normal form

$$-9(x-3) - (y+1) + 5(z-2) = 0$$

$$-9x - y + 5z + 16 = 0$$

$$9x + y - 5z - 16 = 0$$



Example Determine whether the line ℓ : x = 3 + 8t, y = 4 + 5t, z = -3 - tis parallel to the plane x - 3y + 5z = 12.

 $\mathbf{v} = \langle 8, 5, -1 \rangle$ $\mathbf{n} = \langle 1, -3, 5 \rangle$



Example Determine whether the line $\ell: x = 3 + 8t$, y = 4 + 5t, z = -3 - tis parallel to the plane x - 3y + 5z = 12. $\mathbf{v} = \langle 8, 5, -1 \rangle$ $\mathbf{n} = \langle 1, -3, 5 \rangle$ $\mathbf{n} \cdot \mathbf{v} = (1)(8) + (-3)(5) + (5)(-1) = 12 \neq 0$

 \therefore The line and the plane are not parallel.

 \therefore The line and the plane **intersects**.



Example Find the intersection of the line

$$\ell: x = 3 + 8t$$
, $y = 4 + 5t$, $z = -3 - t$
and the plane $x - 3y + 5z = 12$.

Suppose the point of intersection is (x_0, y_0, z_0)



Two distinct intersecting planes determine two positive angles of intersection



If \mathbf{n}_1 and \mathbf{n}_2 are normals to the planes, then the acute angle θ between the planes satisfies:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

Example Find the acute angle of intersection between the two planes

$$4x + 2y + 2z = 6 \text{ and } x + 2y - z = 4$$

$$\mathbf{n}_1 = \langle 4, 2, 2 \rangle \qquad \mathbf{n}_2 = \langle 1, 2, -1 \rangle$$

$$\|\mathbf{n}_1\| = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24} = 2\sqrt{6} \qquad \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{||\mathbf{n}_1|| ||\mathbf{n}_2||}$$

$$\|\mathbf{n}_2\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6} \qquad \cos \theta = \frac{6}{2\sqrt{6} \times \sqrt{6}} = \frac{1}{2}$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (4)(1) + (2)(2) + (2)(-1) = 6$$

θ

 $= \frac{1}{3}$

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6$$
 and $6x + 2y - 3z = 4$



v || Plane 1 ⇒ **v** ⊥ **n**1
v || Plane 2 ⇒ **v** ⊥ **n**2
∴ **v** = **n**1 × **n**2 =
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 6 & 2 & -3 \end{vmatrix}$$

 $\mathbf{v} = \langle 4, 30, 28 \rangle$

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6$$
 and $6x + 2y - 3z = 4$

 $\mathbf{v} = \langle 4, 30, 28 \rangle$

To find a point on ℓ

- ℓ is not perpendicular to $\mathbf{k} = \langle 0, 0, 1 \rangle$
- $\mathbf{v} \cdot \mathbf{k} = 0 + 0 + 28 \neq 0$
- $\therefore \ell$ intersects the xy –plane (z = 0)

$$2x - 4y = 6$$
$$6x + 2y = 4$$

Solve the equations:

- x = 1, y = -1
- : point = (1, -1, 0)

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6$$
 and $6x + 2y - 3z = 4$

 $\mathbf{v} = \langle 4, 30, 28 \rangle$

: point = (1, -1, 0)

The parametric equations of ℓ are

$$x = 1 + 4t$$
$$y = -1 + 30t$$
$$z = 28t$$

DISTANCE PROBLEMS INVOLVING PLANES

- The distance between a point and a plane.
- The distance between two parallel planes.
- The distance between two skew lines.





DISTANCE PROBLEMS INVOLVING PLANES

The distance *D* between a point $P_0(x_0, y_0, z_0)$ and the plane ax + by + cz + d = 0 is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example Find the distance *D* between the point (1, -4, -3) and the plane 2x - 3y + 6z = -1.

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.7] QUADRIC SURFACES Course: Calculus (3)

<u>Chapter: [11]</u> THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.8] CYLINDRICAL AND SPHERICAL COORDINATES

REVIEW OF POLAR COORDINATES



REVIEW OF POLAR COORDINATES



CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS



CONSTANT SURFACES

In rectangular coordinates



CONSTANT SURFACES

In cylindrical coordinates



CONSTANT SURFACES

In spherical coordinates



| From | Cylindrical | | From | Spherical | | From | Spherical | |
|------------|---|----------------|------------|--|----------------|------------|--|-----------------|
| То | Rectangular | | То | Cylindrical | | То | Rectangular | |
| | $x = r \cos \theta$ $y = r \sin \theta$ $z = z$ | | | $r = \rho \sin \phi$ $\theta = \theta$ $z = \rho \cos \phi$ | | | $x = \rho \sin \phi \phi$ $y = \rho \sin \phi s$ $z = \rho \cos \phi$ | cosθ sinθ |
| | | | | | | | | |
| From | Rectangular | | From | Cylindrical | | From | Rectangular | |
| From To | Rectangular Cylindrical | | From To | Cylindrical Spherical | | From To | Rectangular Spherical | |
| From To | Rectangular Cylindrical $r = \sqrt{x^2 + y}$ | , ² | From To | Cylindrical Spherical $\rho = \sqrt{r^2 + z}$ | z ² | From To | Rectangular Spherical $\rho = \sqrt{x^2 + y}$ | $v^{2} + z^{2}$ |
| From To | Rectangular Cylindrical $r = \sqrt{x^2 + y}$ $\tan \theta = y/x$ | <u>,</u> 2 | From To | CylindricalSpherical $\rho = \sqrt{r^2 + 2}$ $\theta = \theta$ | z ² | From To | Rectangular Spherical $\rho = \sqrt{x^2 + y}$ $\tan \theta = y/x$ | $v^2 + z^2$ |

Example Find the rectangular coordinates of the point with cylindrical coordinates

$$(r,\theta,z) = \left(4,\frac{\pi}{3},-3\right)$$

$$x = 4\cos\frac{\pi}{3} = 2$$
$$y = 4\sin\frac{\pi}{3} = 2\sqrt{3}$$
$$z = -3$$
$$\therefore (x, y, z) = (2, 2\sqrt{3}, -3)$$

| Cylindrical |
|---|
| Rectangular |
| $x = r \cos \theta$ $y = r \sin \theta$ |
| |

Example Find the rectangular coordinates of the point with spherical coordinates

$$(\rho,\theta,\phi) = \left(4,\frac{\pi}{3},\frac{\pi}{4}\right)$$

$$x = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{2}{\sqrt{2}} = \sqrt{2}$$
$$y = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$$
$$z = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$
$$\therefore (x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$$

| From | Spherical |
|------|-------------|
| То | Rectangular |

 $x = \rho \sin \phi \cos \theta$ $y = \rho \sin \phi \sin \theta$ $z = \rho \cos \phi$

Example Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = (4, -4, 4\sqrt{6})$$

$$\rho = \sqrt{4^2 + (-4)^2 + (4\sqrt{6})^2} = \sqrt{128} = 8\sqrt{2}$$
$$\tan \theta = \frac{-4}{4} = -1$$
$$\cos \phi = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$

| From | Rectangular | |
|------|-------------|--|
| То | Spherical | |
| | / | |

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = y/x$$

$$\cos \phi = z/\rho$$

Example Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = (4, -4, 4\sqrt{6})$$

$$\rho = \sqrt{4^2 + (-4)^2 + (4\sqrt{6})^2} = \sqrt{128} = 8\sqrt{2}$$

$$\tan \theta = \frac{-4}{4} = -1 \qquad \theta = \frac{7\pi}{4}$$

$$\cos \phi = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2} \qquad \phi = \frac{\pi}{6}$$

$$\therefore (\rho, \theta, \phi) = \left(8\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6}\right)$$


$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta} + \rho^2 \sin^2 \phi \sin^2 \theta$$







Example Find equations of the paraboloid $\rho = \cos \phi \csc^2 \phi$ in cylindrical coordinates.

$$\rho = \cos \phi \csc^2 \phi$$
$$\sin^2 \phi \rho = \cos \phi$$
$$\frac{r^2}{\rho^2} \rho = \frac{z}{\rho}$$
$$z = r^2$$

FromSphericalToCylindrical $r = \rho \sin \phi$ $\theta = \theta$ $z = \rho \cos \phi$



