Course: Calculus (4)

Chapter: [14]

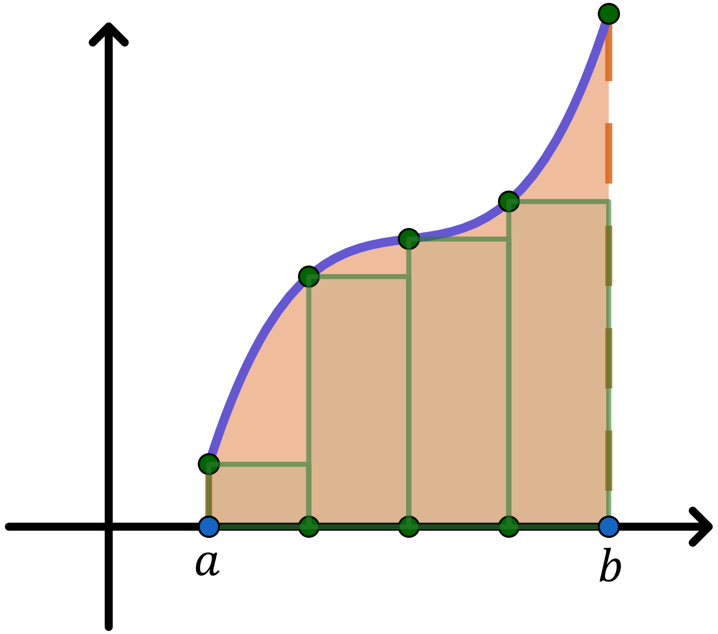
MULTIPLE INTEGRALS

Section: [14.1]

DOUBLE INTEGRALS

THE AREA PROBLEM

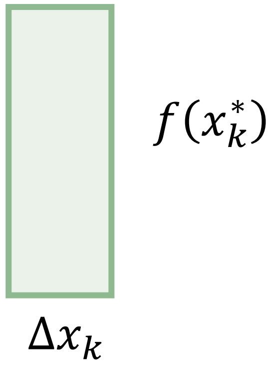
Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x −axis.



$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

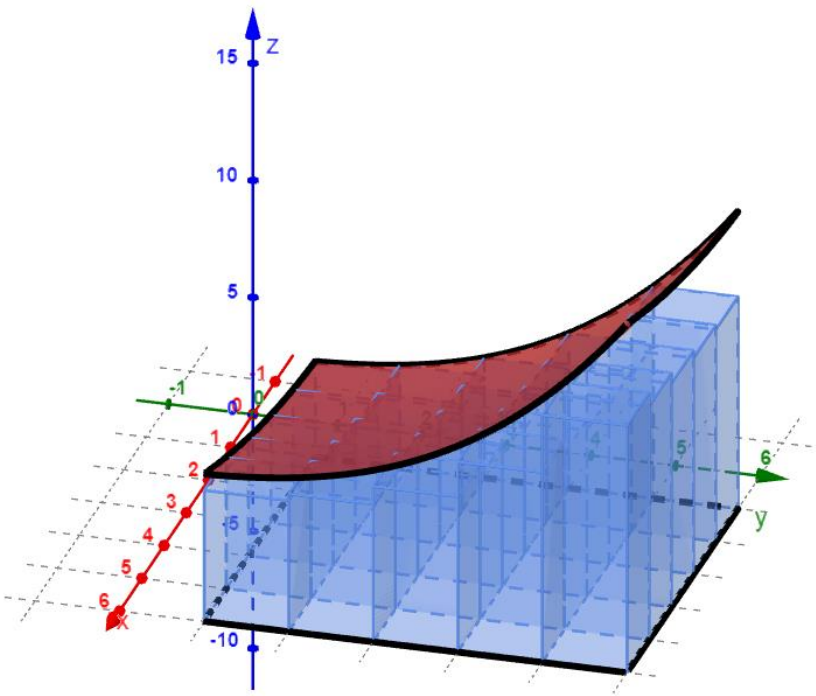
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



THE VOLUME PROBLEM

Given a function f of two variables that is continuous and nonnegative on a region R in the xy −plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R.



We approximate the volume by using rectangular parallelepipeds.

Δy_j

Δx_i $\boxed{\Delta A_{ij} = \Delta x_i \Delta y_j}$

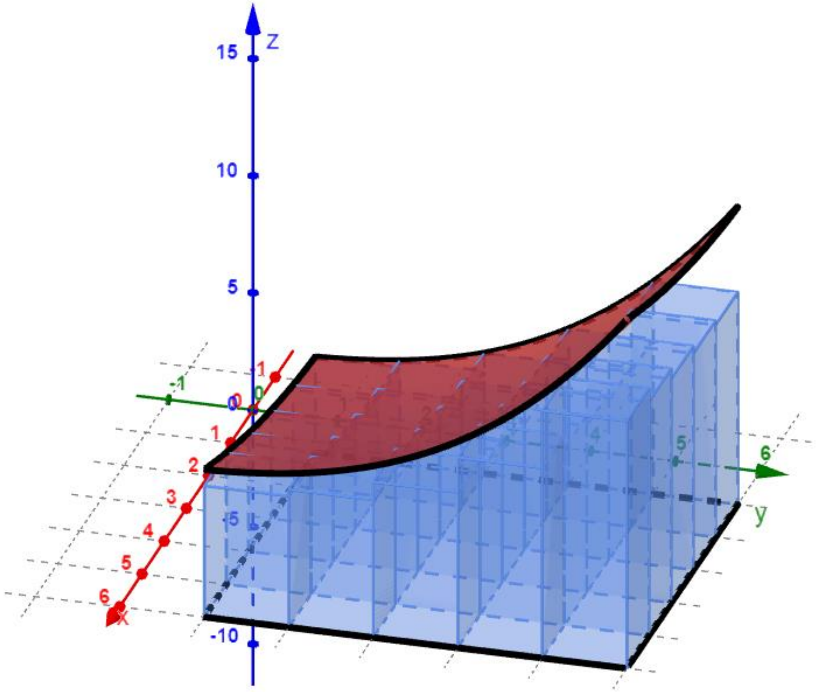
$$V_{\text{box}} = \text{base area} \times \text{height}$$
$$= \Delta A_{ij} \times \; f(x_i^*, y_j^*)$$

THE VOLUME PROBLEM

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta A_{ij}$$

$$V = \lim_{\substack{n \to \infty \\ m \to \infty}} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta A_{ij} \right]$$

$$V = \iint_R f(x, y)\, dA$$



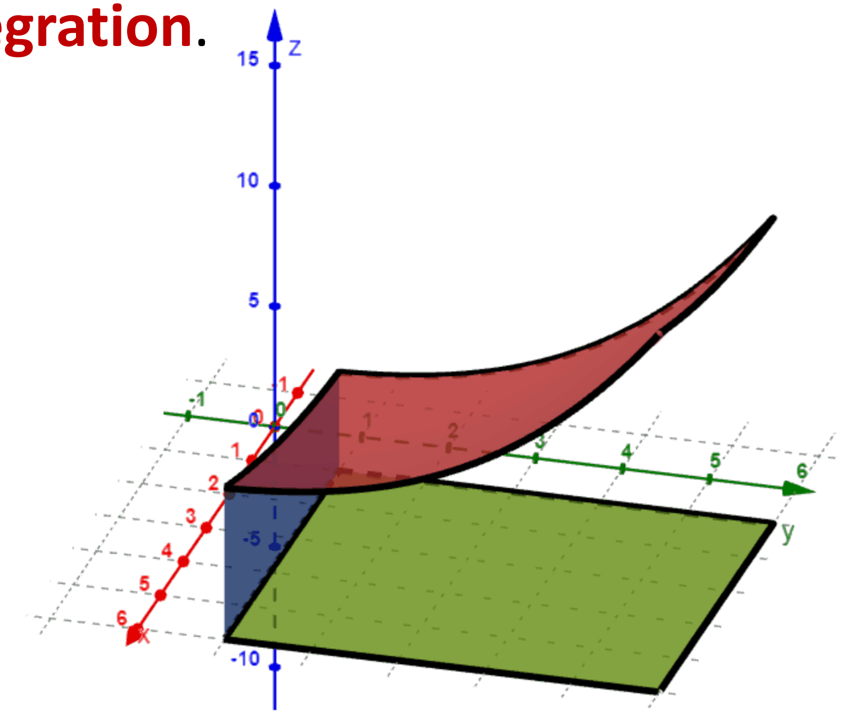
Δy_j

Δx_i $\boxed{\Delta A_{ij} = \Delta x_i \Delta y_j}$

EVALUATING DOUBLE INTEGRALS

- The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable.
- Let us consider the reverse of this process, **partial integration**.

$$\int_a^b f(x,y)dx$$



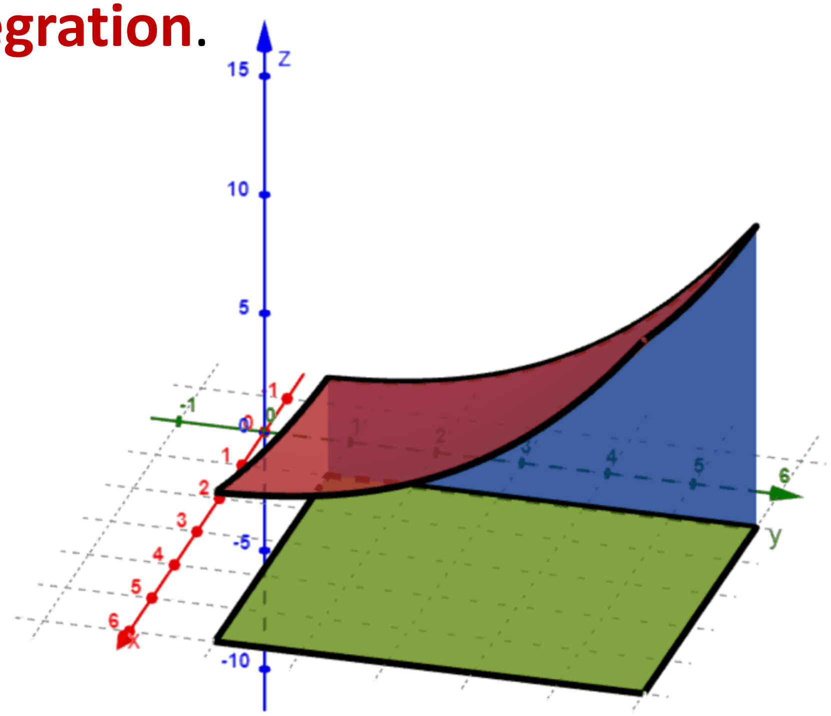
✓ **The partial definite integral with respect to x.**

✓ Is evaluated by holding y fixed and integrating with respect to x.

EVALUATING DOUBLE INTEGRALS

- The partial derivatives of a function $f(x,y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable.
- Let us consider the reverse of this process, **partial integration**.

$$\int_{c}^{d} f(x,y)dy$$



✓ **The partial definite integral with respect to y.**

✓ Is evaluated by holding x fixed and integrating with respect to y.

EVALUATING DOUBLE INTEGRALS

Example **(1)** $\int_0^1 xy^2 dx = y^2 \int_0^1 x dx = \frac{y^2x^2}{2}\Big]_0^1 = \frac{y^2}{2}$

(2) $\int_0^1 xy^2 dy = x \int_0^1 y^2 dy = \frac{xy^3}{3}\Big]_0^1 = \frac{x}{3}$

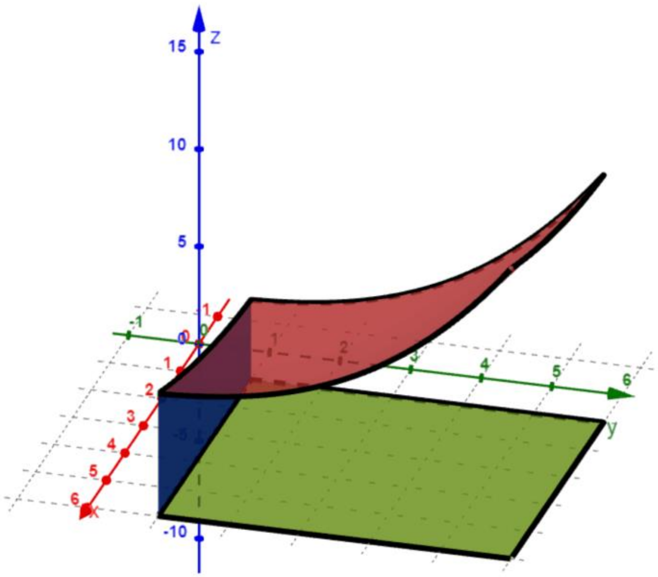
NOTE

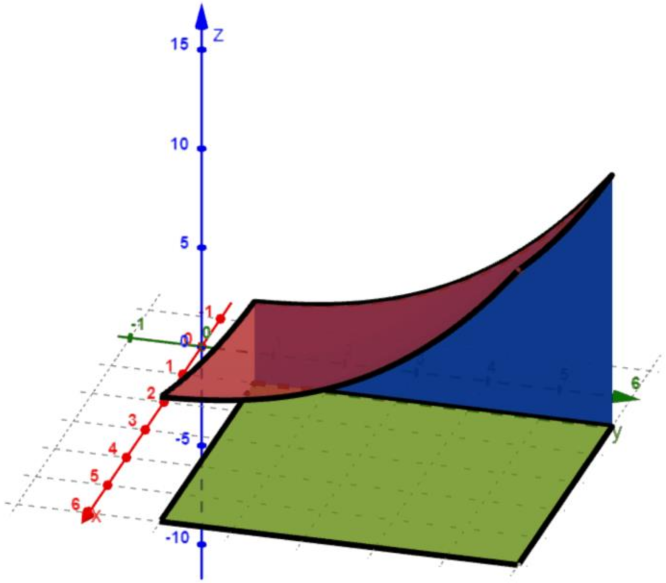
- A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y.
- A partial definite integral with respect to y can be integrated with respect to x.
- This two-stage integration process is called **iterated** (or *repeated*) **integration**.

EVALUATING DOUBLE INTEGRALS

$$\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

$$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$





EVALUATING DOUBLE INTEGRALS

- We introduce the following notation:

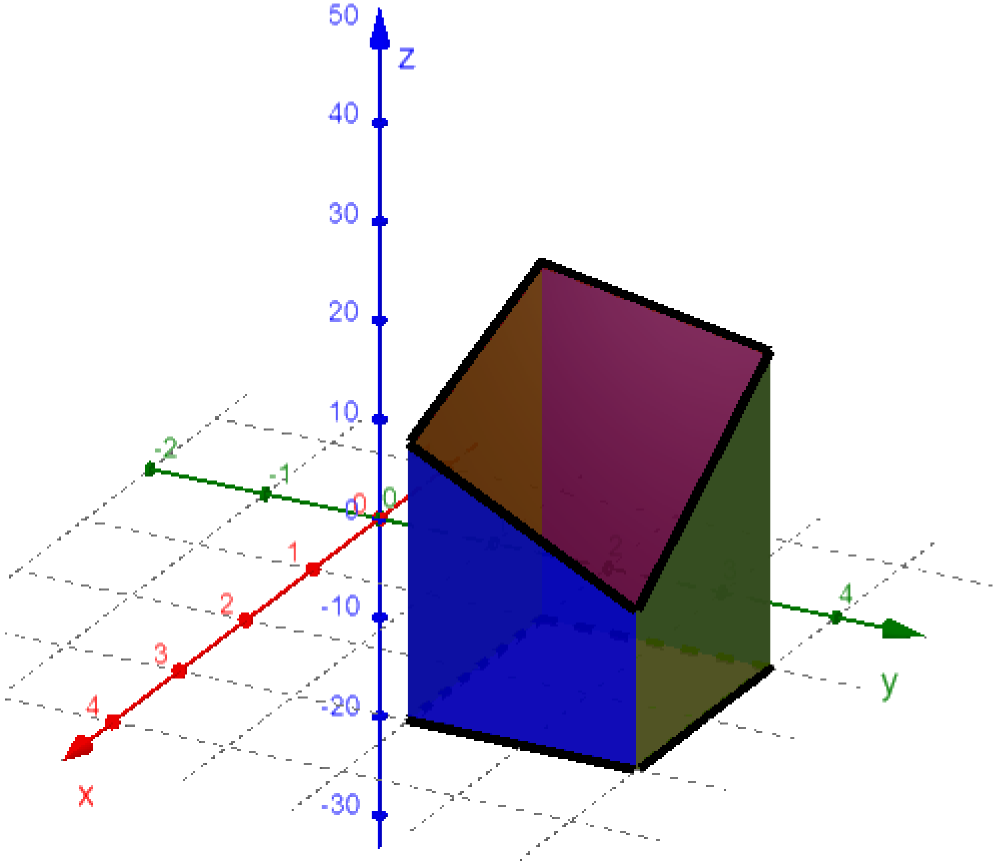
$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

- These integrals are called *iterated integrals*.

EVALUATING DOUBLE INTEGRALS

Example Evaluate $\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx$



$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx$$

$$= \int_1^3 (40y - xy^2)\Big]_2^4\,dx$$

$$= \int_1^3 [(160 - 16x) - (80 - 4x)]\,dx = \int_1^3 (80 - 12x)\,dx$$

$$= 112$$

EVALUATING DOUBLE INTEGRALS

Homework Evaluate $\displaystyle\int_2^4\int_1^3 (40-2xy)\,dx\,dy \quad = 112$

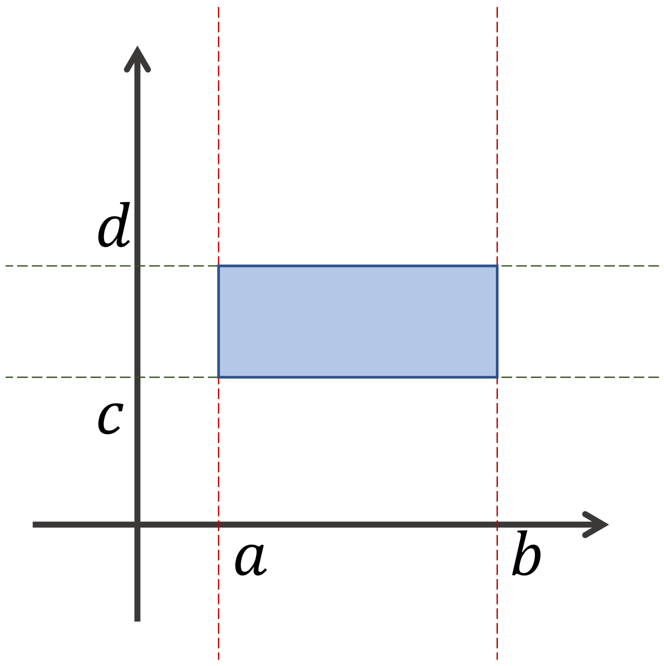
Fubini's Theorem

Let R be the rectangle defined by

$$R = \{(x,y) : \ a \le x \le b\ , c \le y \le d\}$$
$$= [a,b] \times [c,d]$$

If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$



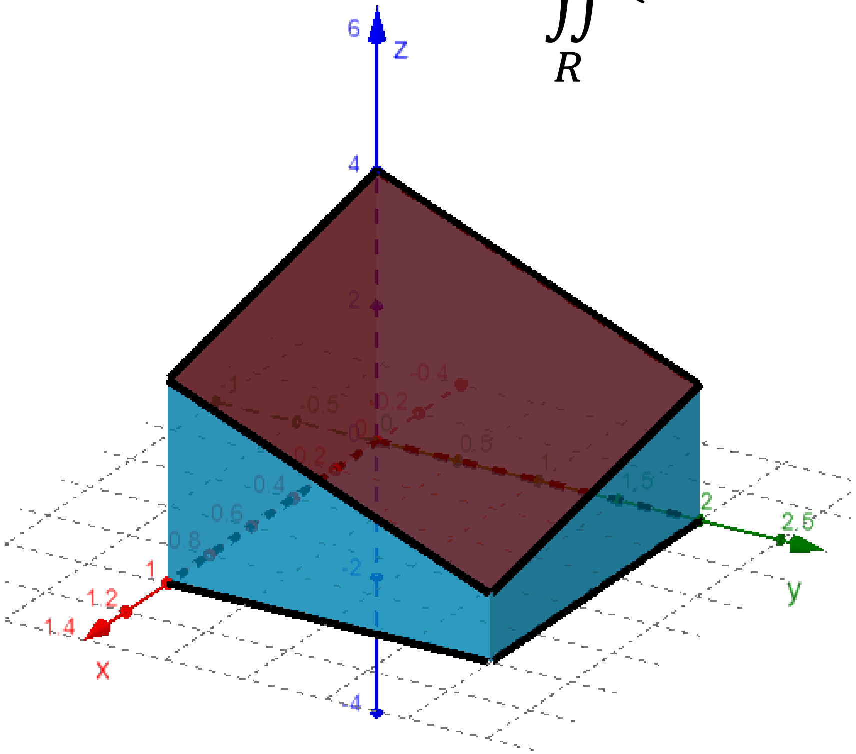
EVALUATING DOUBLE INTEGRALS

Example Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0,1] \times [0,2]$.

$$V = \iint_R (4 - x - y)\,dA = \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx = \int_0^1 \left[\int_0^2 (4 - x - y)\,dy\right] dx$$

$$= \int_0^1 \left(4y - xy - \frac{y^2}{2}\right)\Bigg]_0^2 dx$$

$$= \int_0^1 (6 - 2x)\,dx \quad = 5 \quad \boxed{= \int_0^2 \int_0^1 (4 - x - y)\,dx\,dy}$$

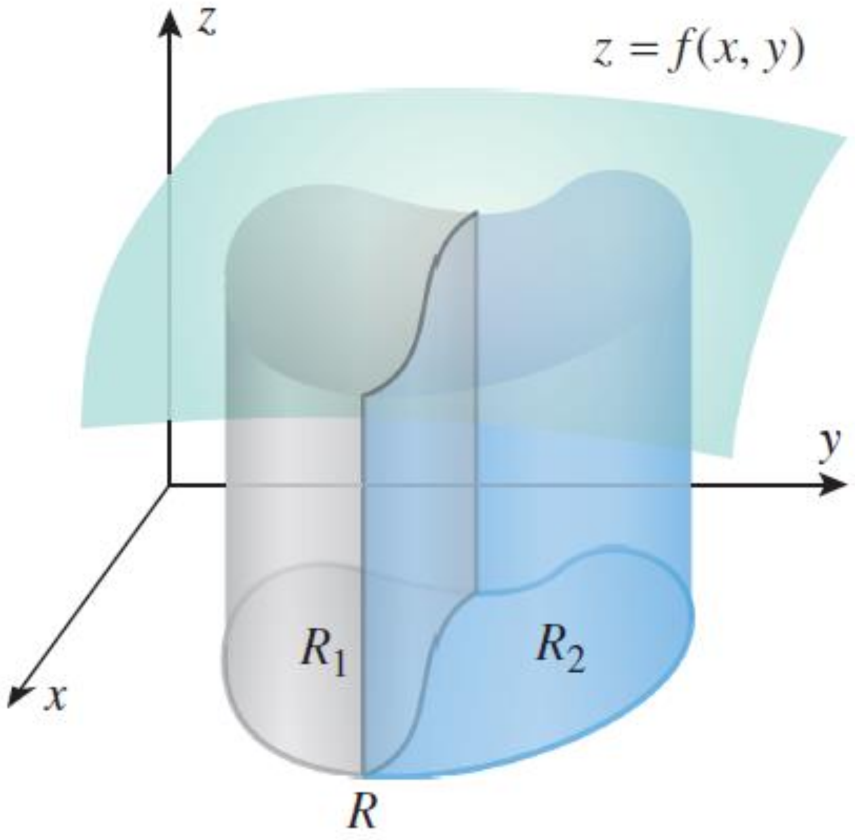


PROPERTIES OF DOUBLE INTEGRALS

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \qquad (c \text{ constant})$$

$$\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$



PROPERTIES OF DOUBLE INTEGRALS

NOTE If $R = [a, b] \times [c, d]$ is a rectangular region, and $f(x, y) = g(x)h(y)$, then

$$\boxed{\iint_R f(x,y)dA = \iint_R g(x)h(y)dA = \left[\int_a^b g(x)dx\right]\left[\int_c^d h(y)dy\right]}$$

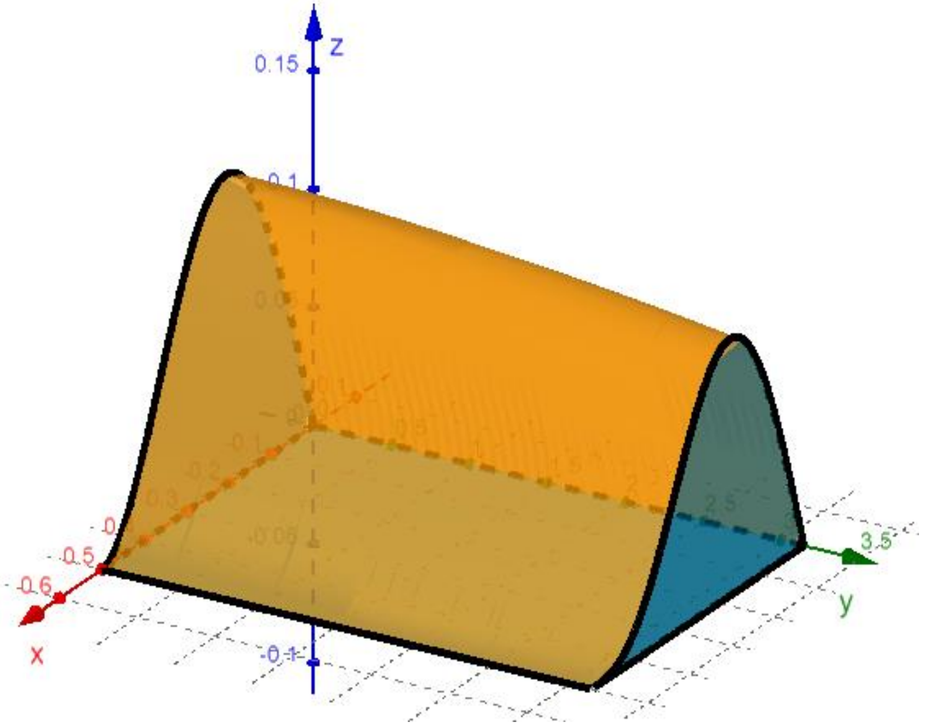
Example $$\int_0^1 \int_0^2 e^{x+y}\, dxdy = \int_0^1 \int_0^2 e^x e^y\, dxdy$$

$$= \left(\int_0^2 e^x dx\right)\left(\int_0^1 e^y dy\right) = (e^2 - 1)(e - 1)$$

EXERCISE SET 14.1 QUESTION 33

Homework Evaluate the integral by choosing a convenient order of integration:

$$\boxed{\frac{1}{3\pi} =} \iint_R x\cos(xy)\cos^2(\pi x)\, dA \quad ; \quad R = \left[0, \tfrac{1}{2}\right] \times [0, \pi]$$



Course: Calculus (4)

Chapter: [14]

MULTIPLE INTEGRALS

Section: [14.2]

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

In this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals

Example $$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x \, dy \right] dx = \int_0^1 \frac{xy^3}{3} \Bigg]_{-x}^{x^2} dx$$

$$= \int_0^1 \left(\frac{x^7}{3} + \frac{x^4}{3} \right) dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \Bigg]_0^1 = \frac{13}{120}$$

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Example $$\int_0^{\pi/3}\int_0^{\cos y} x\sin y\,dx\,dy = \int_0^{\pi/3}\left[\int_0^{\cos y} x\sin y\,dx\right]dy$$

$$= \int_0^{\pi/3} \left.\frac{x^2\sin y}{2}\right]_0^{\cos y} dy$$

$$= \int_0^{\pi/3} \frac{1}{2}\cos^2 y\sin y\,dy = -\frac{1}{2}\int_1^{1/2} t^2 \sin y \frac{dt}{\sin y}$$

$$= \frac{1}{2}\int_{1/2}^{1} t^2 dt = \left.\frac{t^3}{6}\right]_{1/2}^{1} = \frac{7}{48}$$

By Substitution

Let $t = \cos y$

$$\frac{dt}{dy} = -\sin y$$

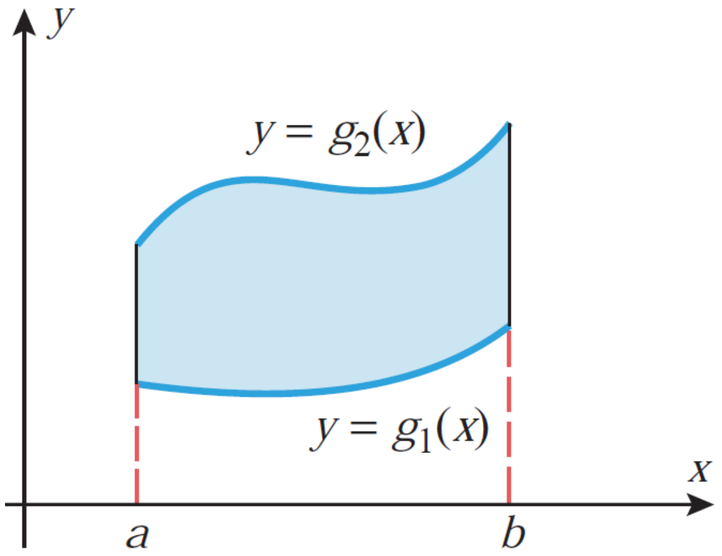
$$dy = -\frac{dt}{\sin y}$$

$y = \pi/3$	$t = 1/2$
$y = 0$	$t = 1$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

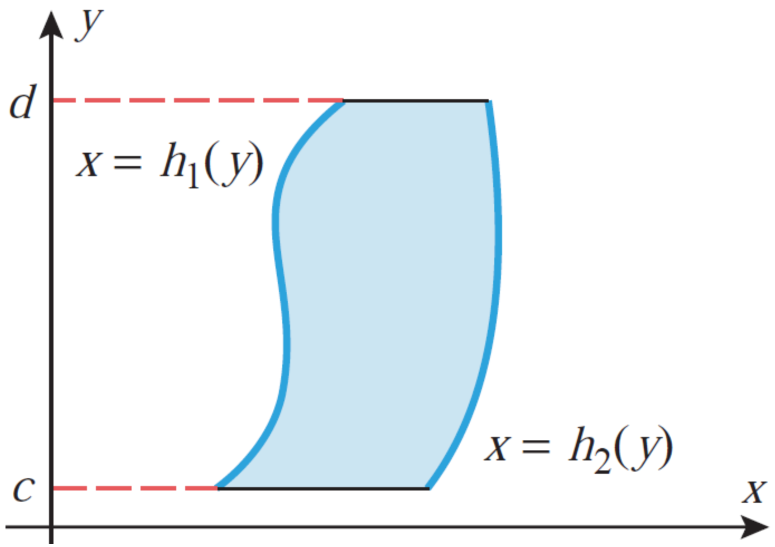
Type I Region

is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.



Type II Region

is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$



DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

1) If R is a **type I region** on which $f(x, y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2) If R is a **type II region** on which $f(x, y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

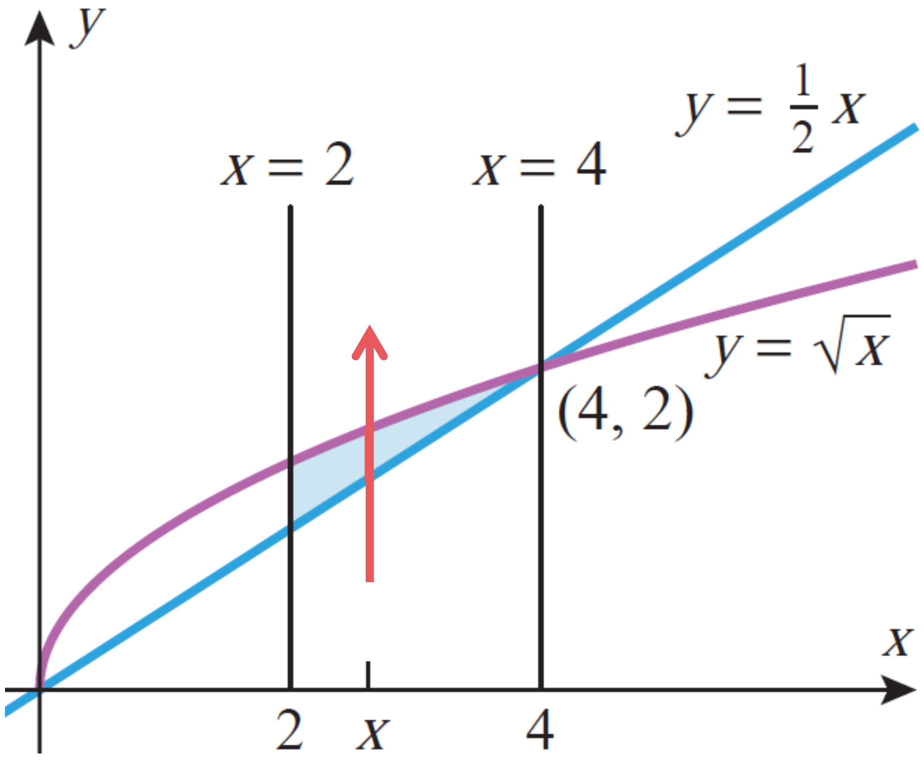
DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint\limits_R xy\,dA$ over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$ and $x = 4$.

Type I Region

$$\iint\limits_R xy\,dA = \int\int xy\,dy\,dx = \int_2^4 \left[\int_{x/2}^{\sqrt{x}} xy\,dy\right] dx$$

$$= \int_2^4 \frac{xy^2}{2}\Bigg]_{x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx = \frac{11}{6}$$



DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

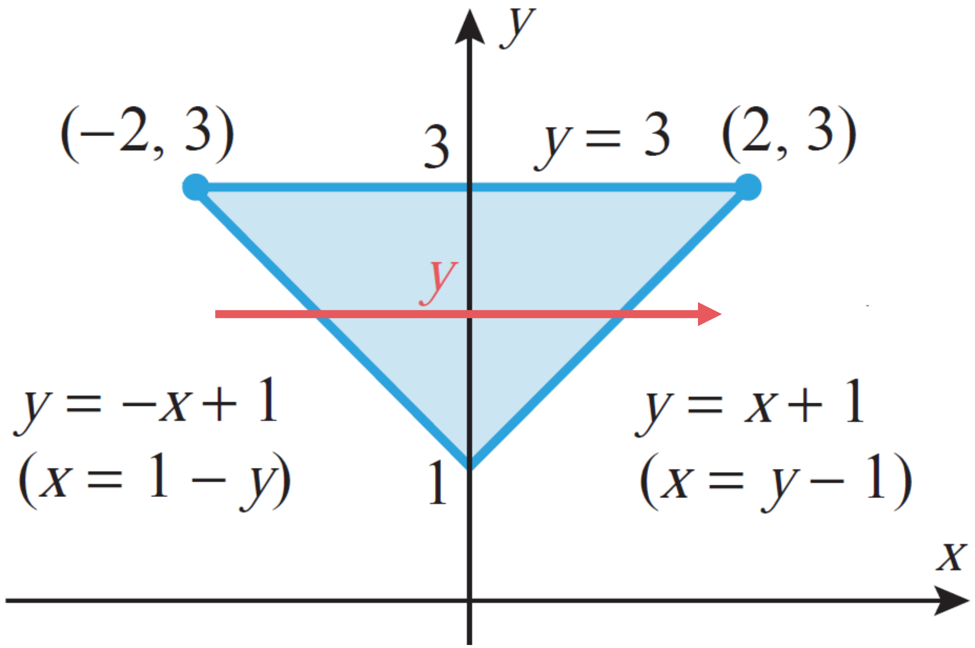
Example Evaluate $\iint\limits_R (2x - y^2)\,dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint\limits_R (2x - y^2)\,dA = \int \int (2x - y^2)\,dx\,dy$$

$$= \int_1^3 x^2 - xy^2 \Big]_{1-y}^{y-1} dy$$

$$= \int_1^3 (2y^2 - 2y^3)\,dy = -\frac{68}{3}$$

Type II Region

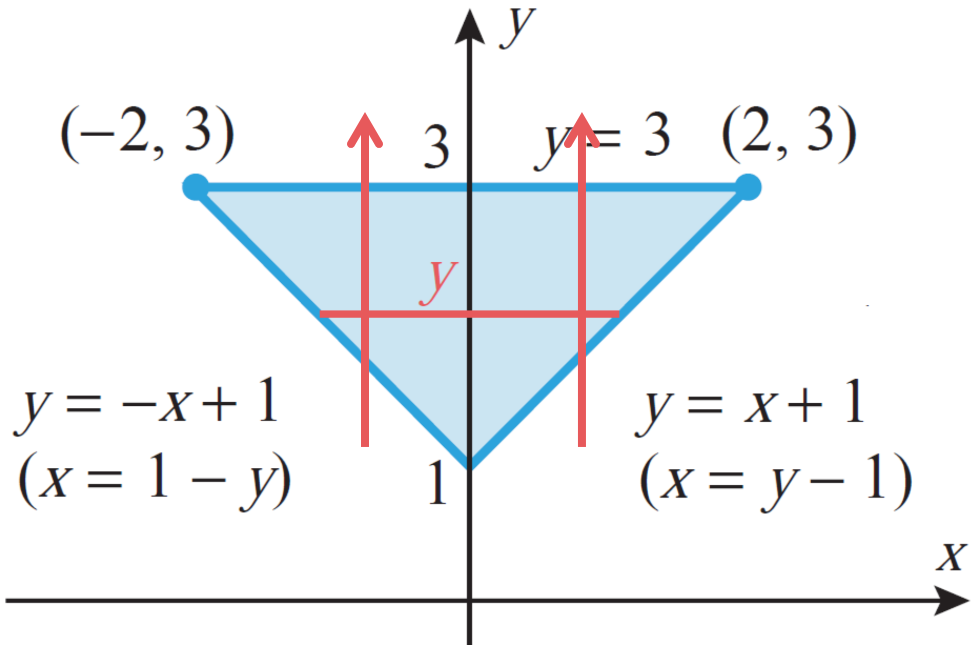


DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint\limits_R (2x - y^2)\,dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint\limits_R (2x - y^2)\,dA$$

Type I Region



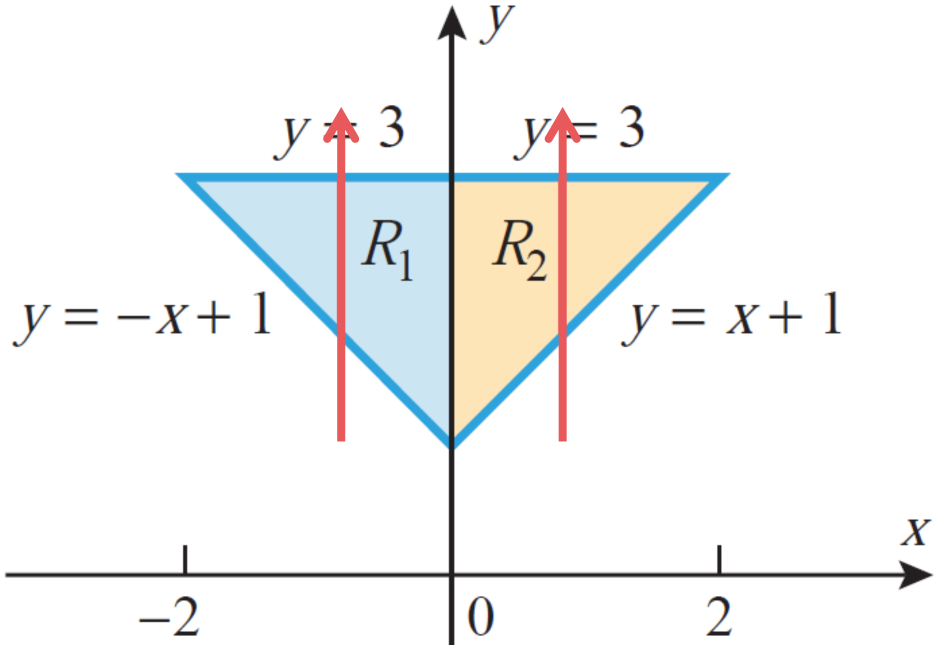
DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint_R (2x - y^2)dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint_R (2x - y^2)dA = \iint_{R_1} (2x - y^2)dA + \iint_{R_2} (2x - y^2)dA$$

$$= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)dydx + \int_{0}^{2} \int_{x+1}^{3} (2x - y^2)dydx$$

Type I Region



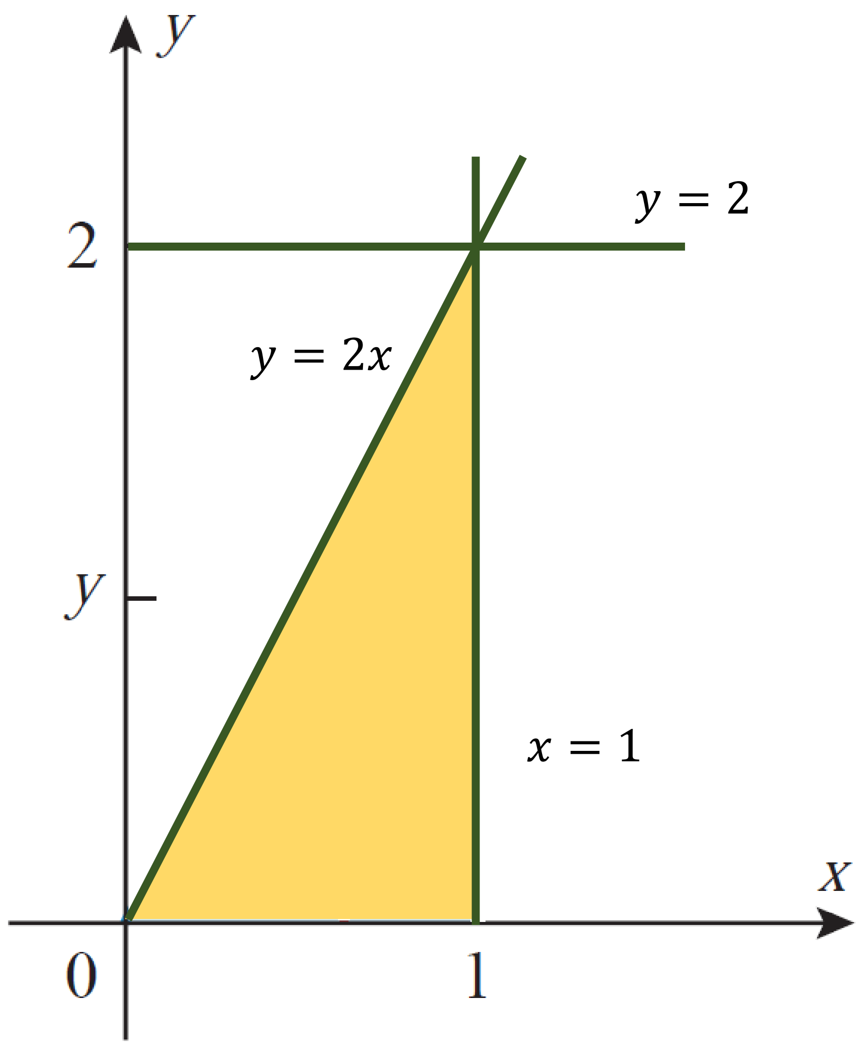
DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\displaystyle\int_0^2 \int_{\boxed{y/2}}^1 e^{x^2}\,dxdy$

$\boxed{y/2} \longrightarrow y = 2x$

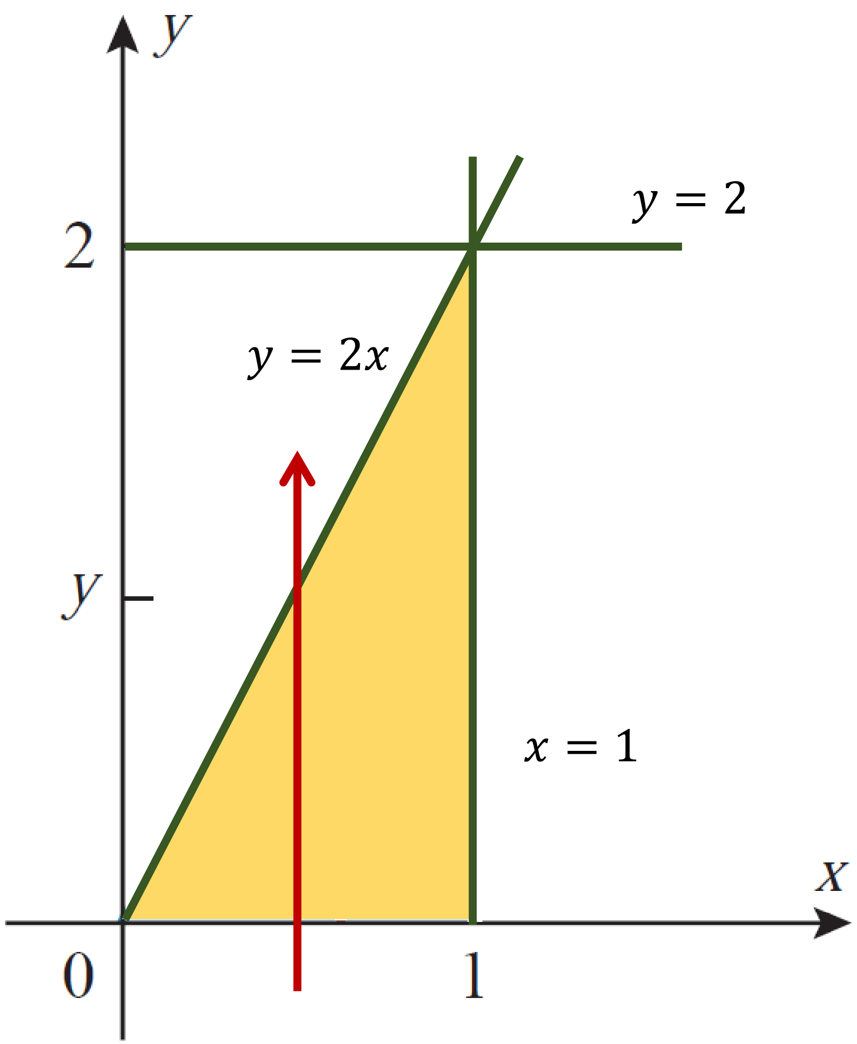
Since there is no elementary antiderivative of e^{x^2}, the integral cannot be evaluated by performing the x −integration first.

We will try to evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.



DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$



$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy = \int \int e^{x^2} dy dx = \int_0^1 \left[\int_0^{2x} e^{x^2} dy \right] dx$$

$$= \int_0^1 e^{x^2} y \Big]_0^{2x} dx$$

By Substitution
Let $t = x^2$

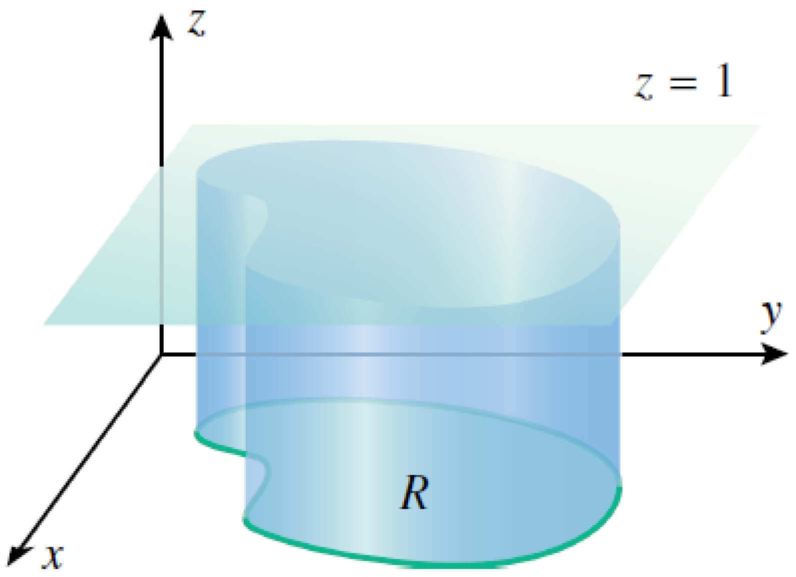
$$= \int_0^1 2x e^{x^2} dx = \int_0^1 e^t dt = e - 1$$

AREA CALCULATED AS A DOUBLE INTEGRAL

Example

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{area of } R = \iint\limits_R 1\, dA = \iint\limits_R dA$$



Cylinder with base R and height 1

AREA CALCULATED AS A DOUBLE INTEGRAL

$$\text{area of } R = \iint_R 1\,dA = \iint_R dA$$

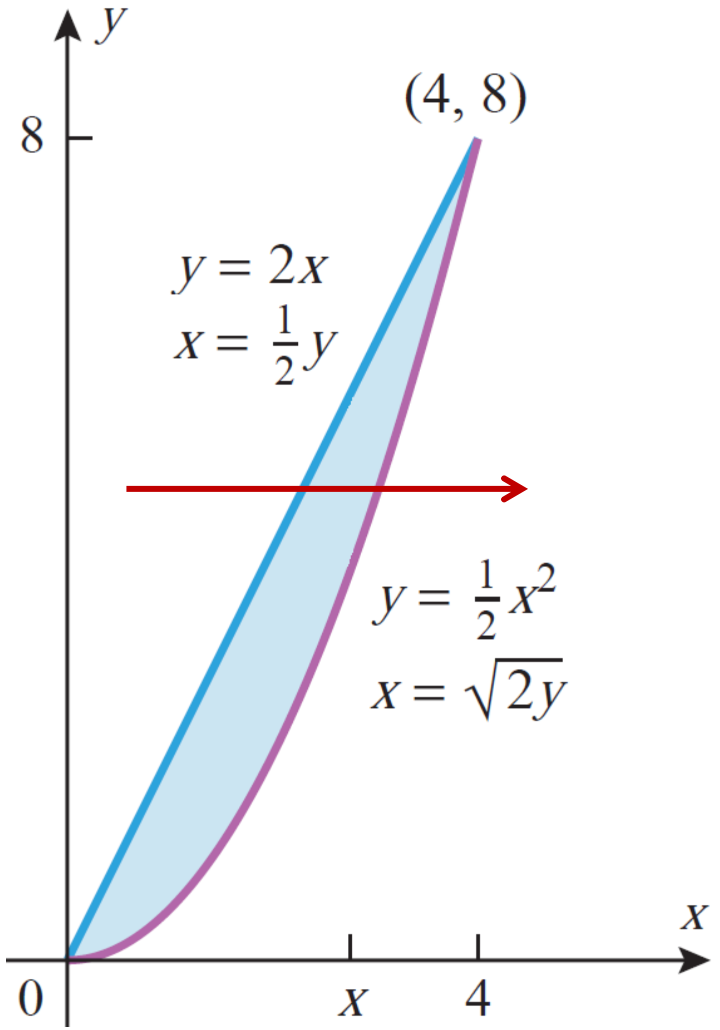
Example Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{Area of } R = \iint_R dA$$

(Type II Region)

$$= \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 x\Big]_{y/2}^{\sqrt{2y}} dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{y}{2}\right) dy = \frac{16}{3}$$



AREA CALCULATED AS A DOUBLE INTEGRAL

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA$$

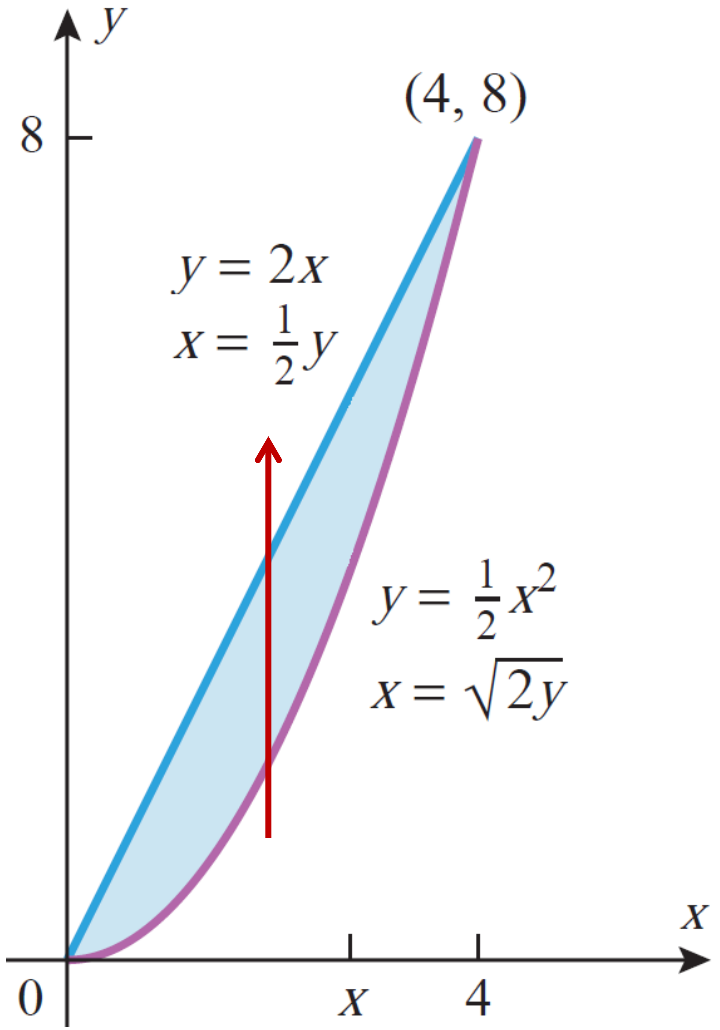
Example Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{Area of } R = \iint_R dA$$

(Type I Region)

$$= \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 y\Big]_{x^2/2}^{2x} dx$$

$$= \int_0^4 \left(2x - \frac{x^2}{2}\right) dx = \frac{16}{3}$$

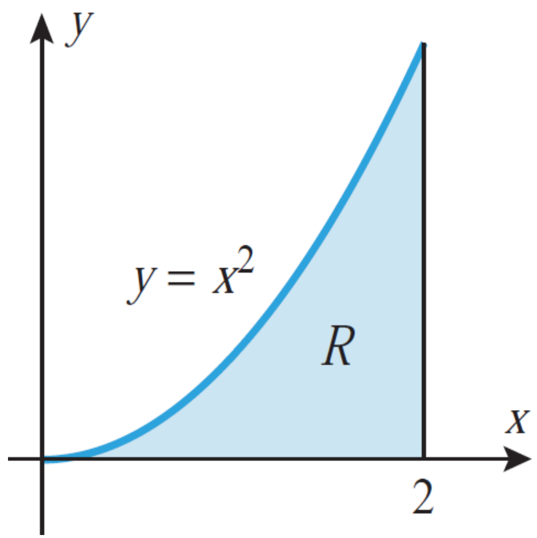


EXERCISE SET 14.2

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$



Course: Calculus (4)

Chapter: [14]

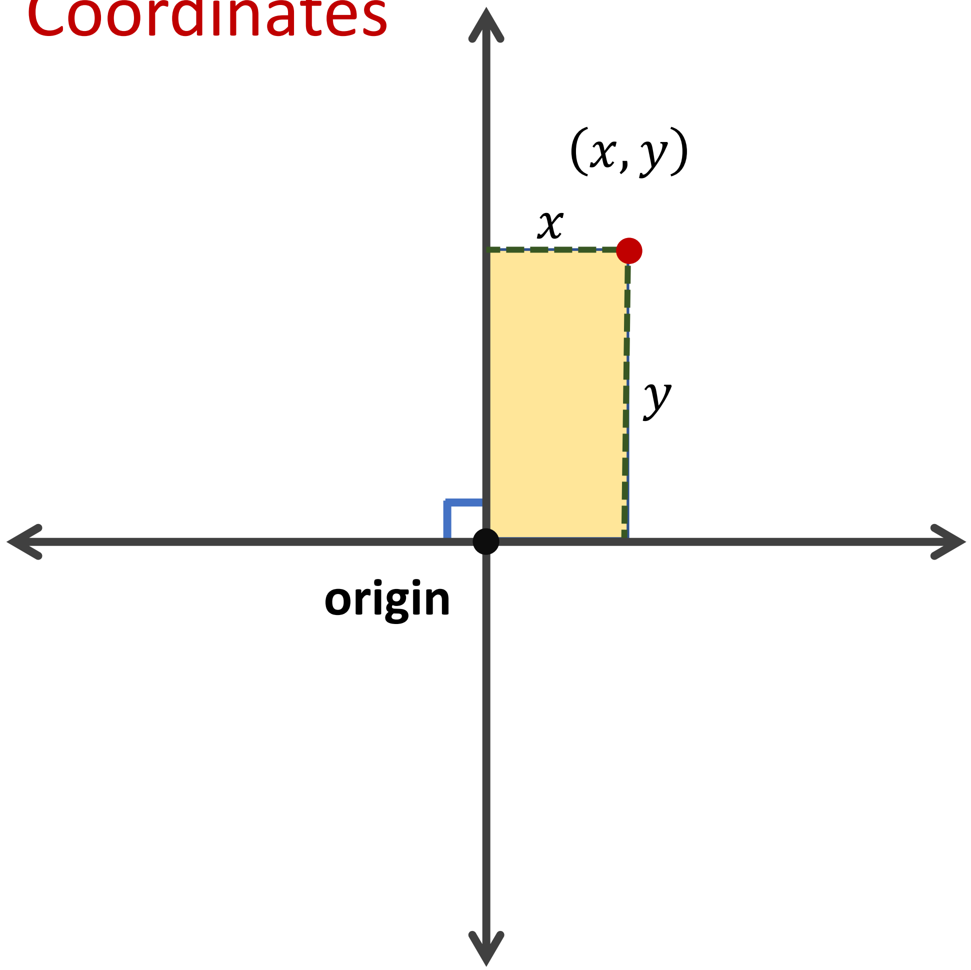
MULTIPLE INTEGRALS

Section: [14.3]

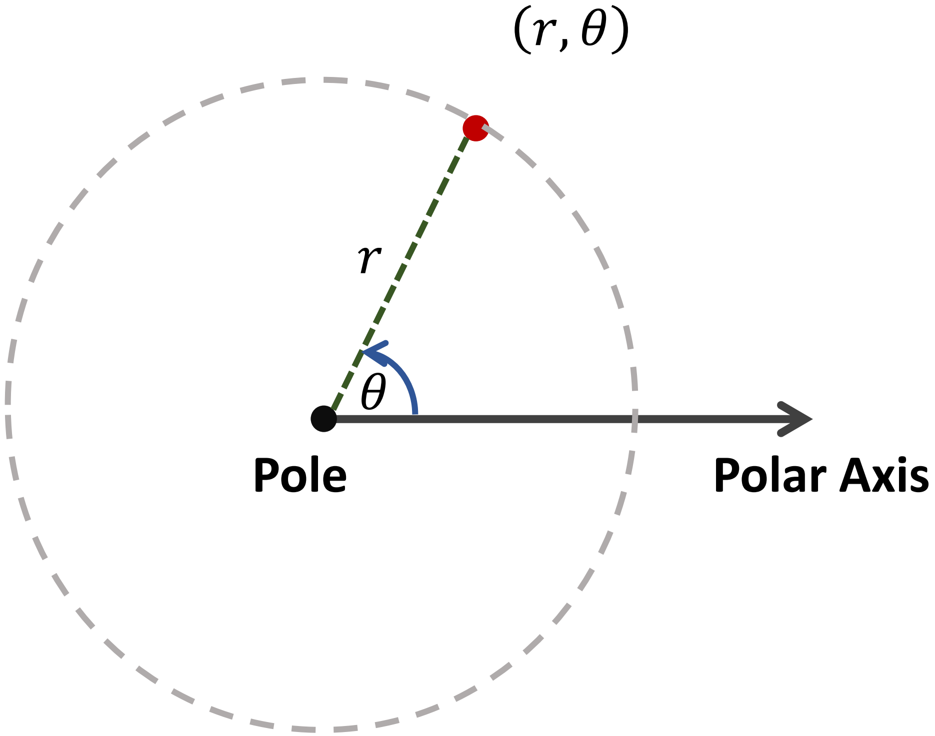
DOUBLE INTEGRALS IN POLAR COORDINATES

REVIEW OF POLAR COORDINATES

Rectangular Coordinates



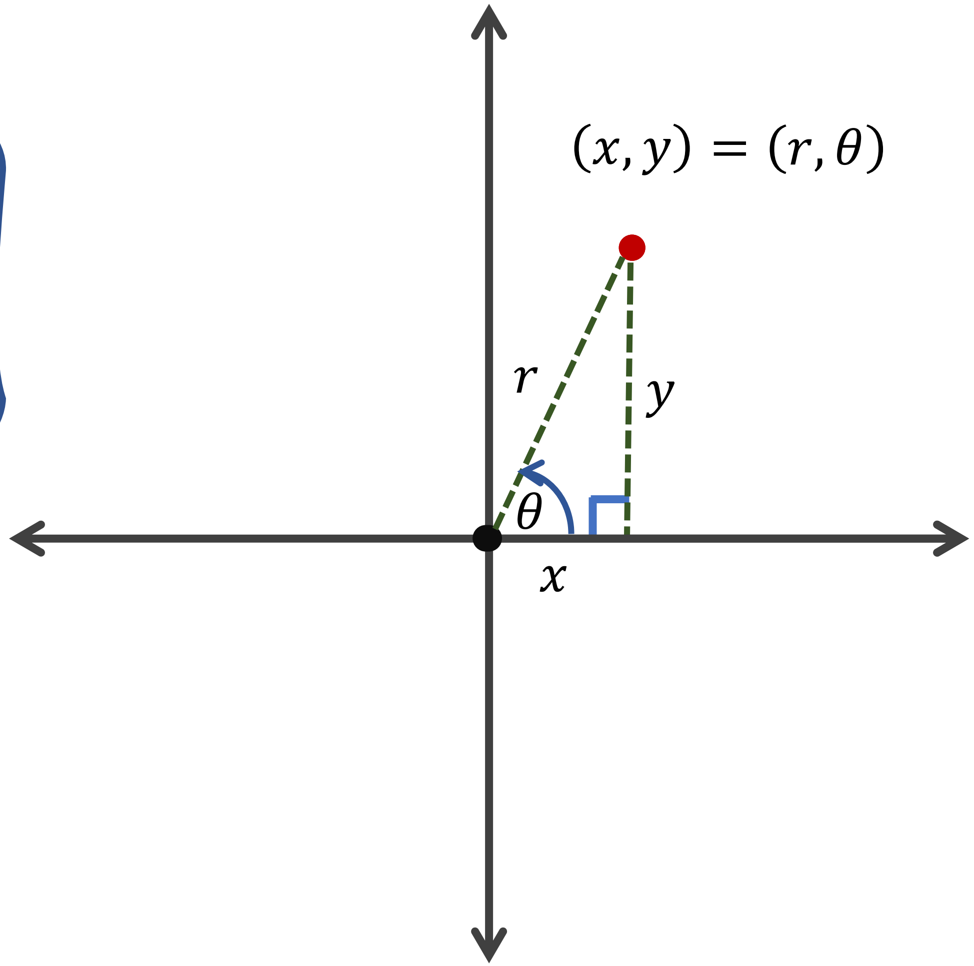
Polar Coordinates



REVIEW OF POLAR COORDINATES

From Rectangular
To Polar

$$\boxed{\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}}$$



From Polar
To Rectangular

$$\cos\theta = \frac{x}{r} \ , \ \sin\theta = \frac{y}{r}$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

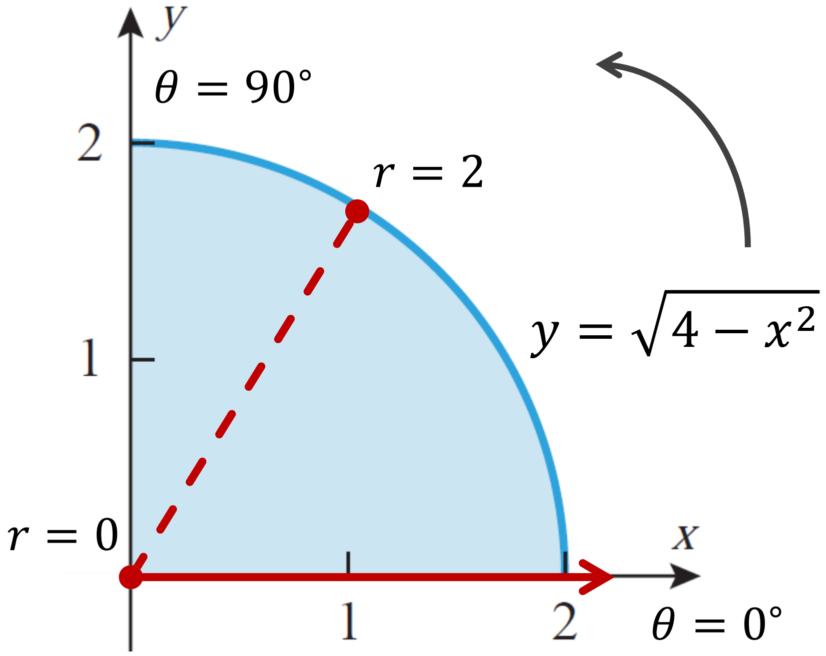
SIMPLE POLAR REGIONS

- Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates.
- This is usually true if the region is bounded by any curve whose equation is simpler in polar coordinates than in rectangular coordinates.
- **Example:** Consider the quarter-disk $x^2 + y^2 = 4$ in the first quadrant shown below.

Rectangular Coordinates

$$0 \le x \le 2$$

$$0 \le y \le \sqrt{4 - x^2}$$



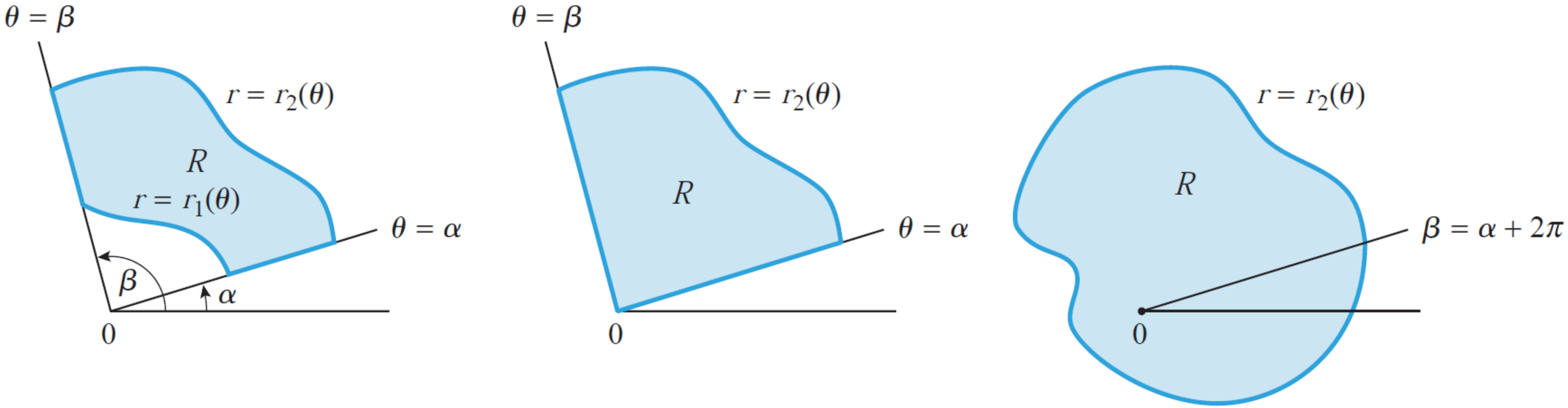
Polar Coordinates

$$0 \le r \le 2$$

$$0 \le \theta \le \pi/2$$

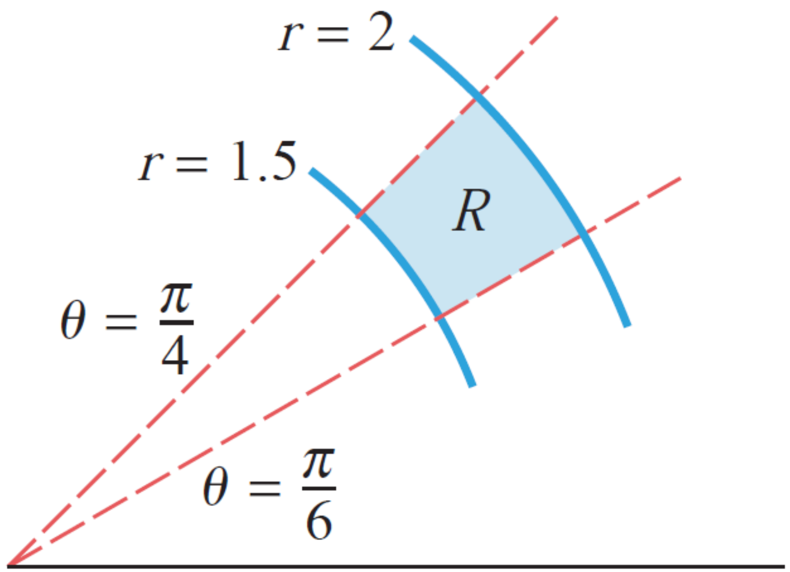
SIMPLE POLAR REGIONS

- Double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.
- The figure below shows a region R in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$.
- If the functions $r_1(\theta)$ and $r_2(\theta)$ are continuous and their graphs do not cross, then the region R is called a *simple polar region*.



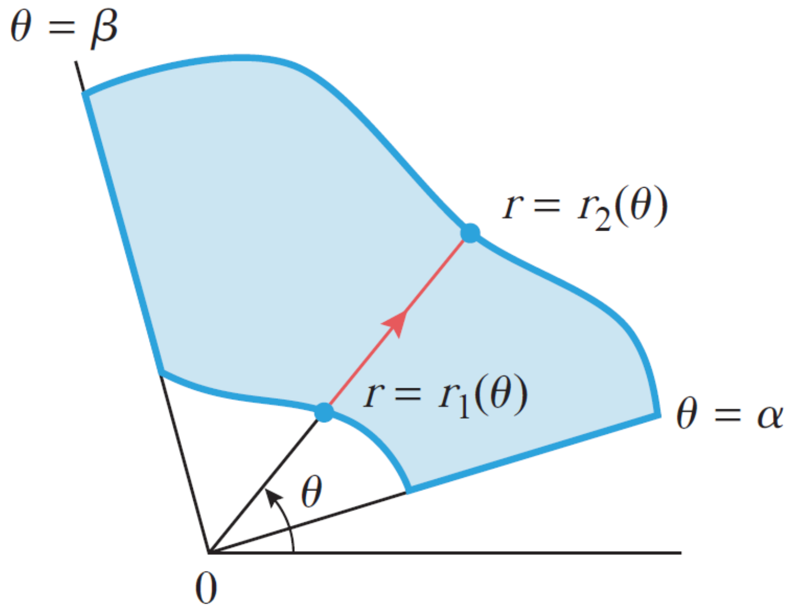
DOUBLE INTEGRALS IN POLAR COORDINATES

NOTE A **polar rectangle** is a simple polar region for which the bounding polar curves are circular arcs.



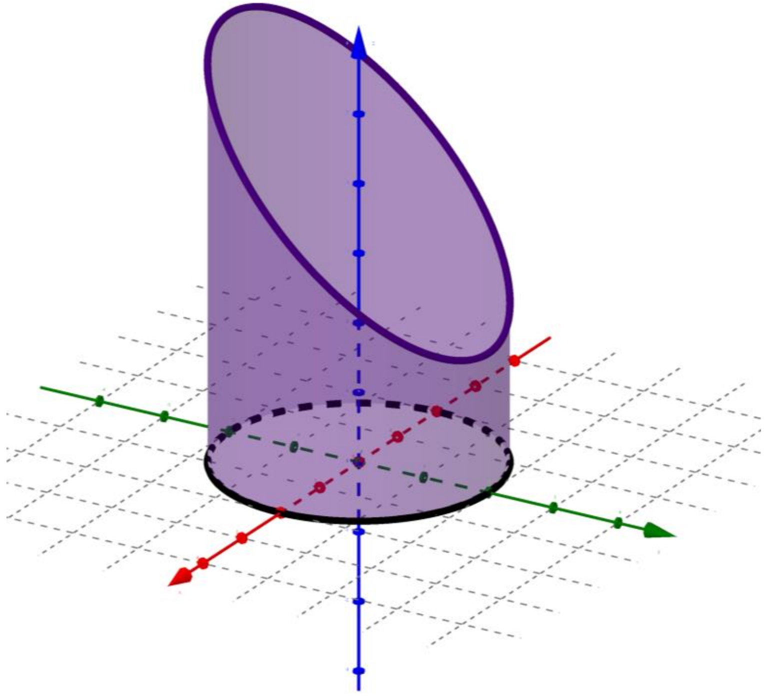
Theorem If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$, and if $f(r, \theta)$ is continuous on R, then

$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r dr d\theta$$



DOUBLE INTEGRALS IN POLAR COORDINATES

Example Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$.



DOUBLE INTEGRALS IN POLAR COORDINATES

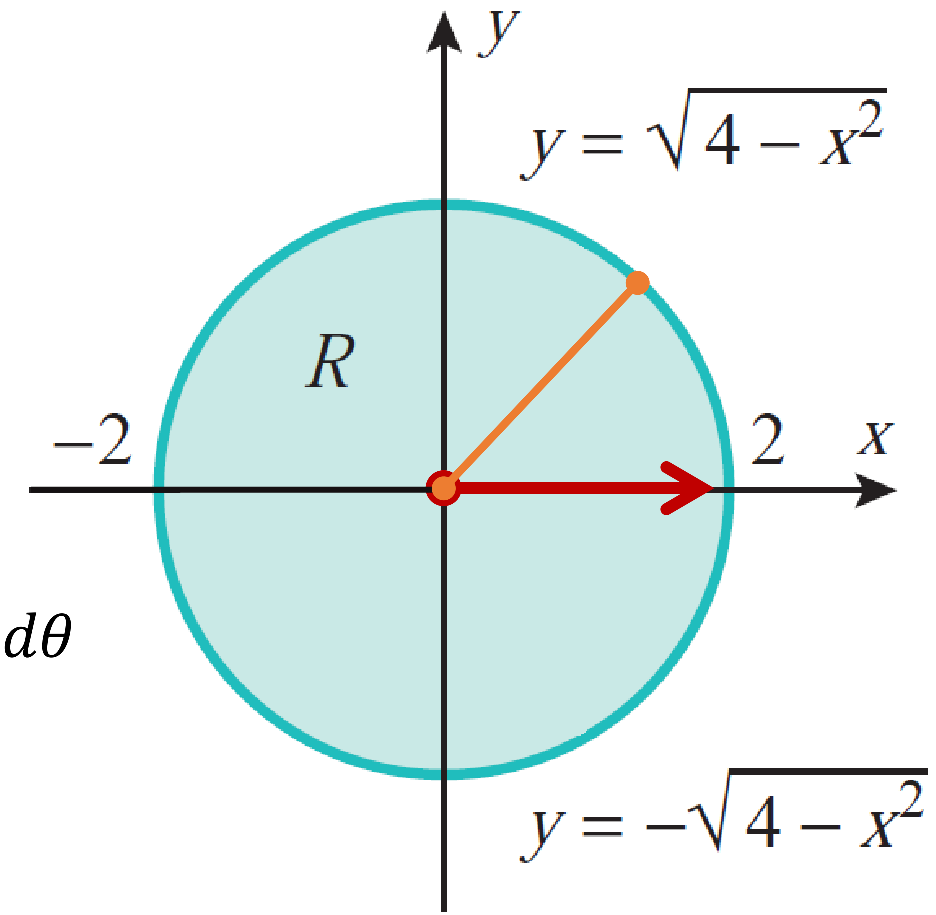
Example Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$.

$$V = \iint_R (4 - y)\,dA = \int \int (4 - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\int_0^2 (4r - r^2 \sin\theta)\,dr \right] d\theta$$

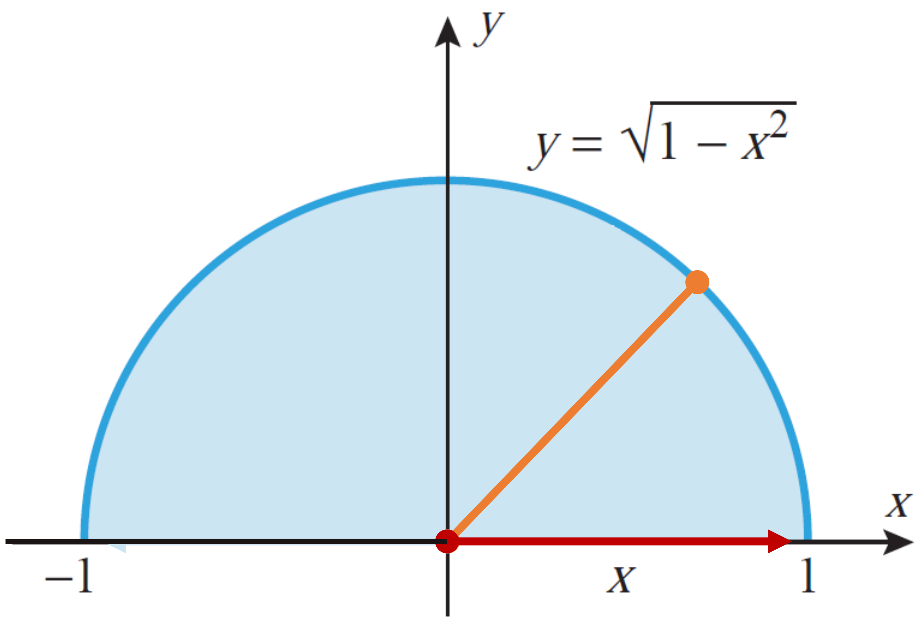
$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{3} r^3 \sin\theta \right) \Bigg]_0^2 d\theta = \int_0^{2\pi} \left(8 - \frac{8}{3}\sin\theta \right) d\theta$$

$$= \left(8\theta + \frac{8}{3}\cos\theta \right) \Bigg]_0^{2\pi} = 16\pi$$



DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

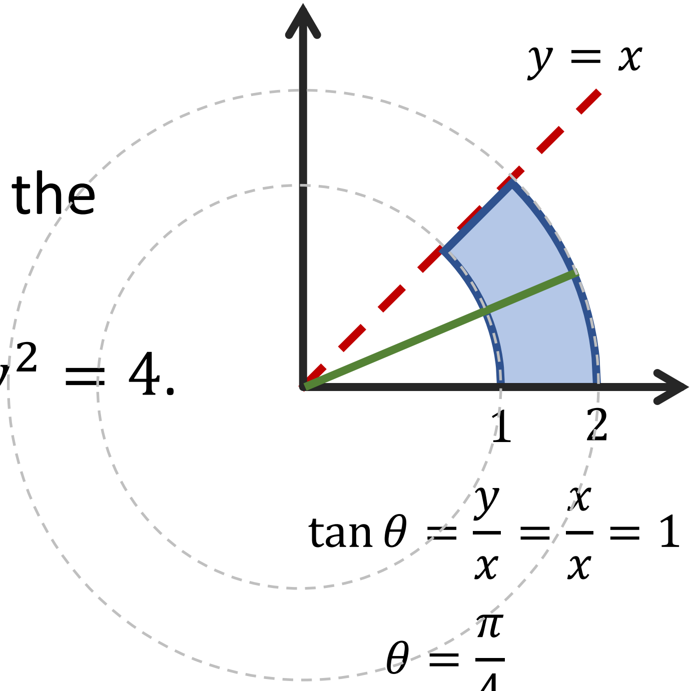


$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int\int (r^2)^{3/2}\,r\,dr\,d\theta$$

$$= \int_{0}^{\pi}\int_{0}^{1} r^4\,dr\,d\theta = \int_{0}^{\pi} \frac{1}{5}\,d\theta = \frac{\pi}{5}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\iint\limits_R \frac{1}{1+x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $y=0$, $y=x$, $x^2+y^2=1$ and $x^2+y^2=4$.



$$\tan\theta = \frac{y}{x} = \frac{x}{x} = 1$$

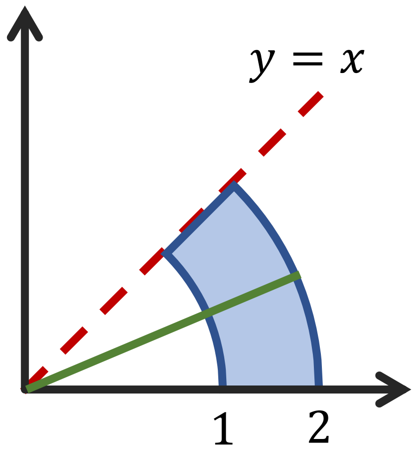
$$\theta = \frac{\pi}{4}$$

$$\iint\limits_R \frac{1}{1+x^2+y^2}\,dA = \int \int \frac{1}{1+r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left[\int_1^2 \frac{r}{1+r^2}\,dr\right] d\theta$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\iint\limits_R \frac{1}{1+x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $y=0$, $y=x$, $x^2+y^2=1$ and $x^2+y^2=4$.



$$\tan\theta = \frac{y}{x} = \frac{x}{x} = 1$$

$$\theta = \frac{\pi}{4}$$

$$\iint\limits_R \frac{1}{1+x^2+y^2}\,dA = \int\int \frac{1}{1+r^2}\,r\,dr\,d\theta$$

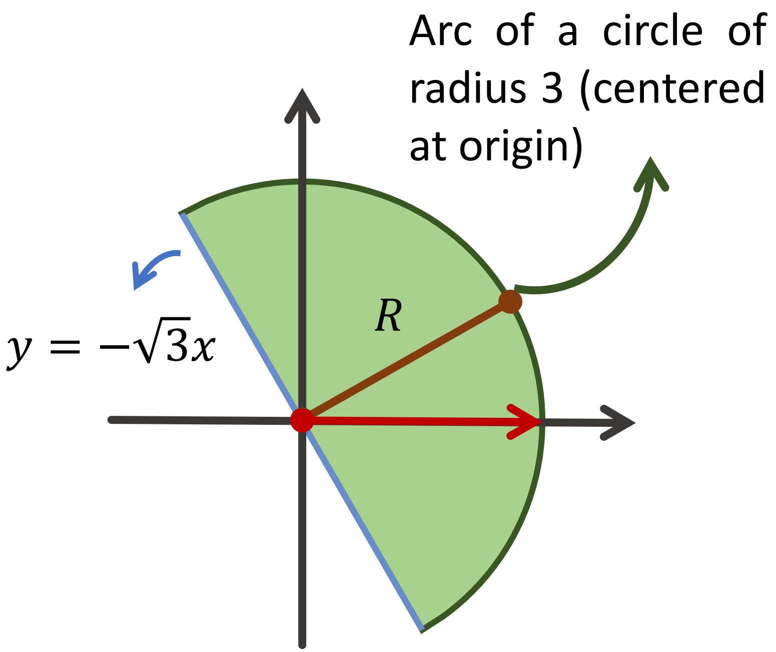
$$= \int_0^{\pi/4}\left[\frac{1}{2}\int_1^2 \frac{2r}{1+r^2}\,dr\right]d\theta = \int_0^{\pi/4} \frac{1}{2}\ln|1+r^2|\Big]_1^2\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\ln\left(\frac{5}{2}\right)d\theta = \frac{\pi}{8}\ln\left(\frac{5}{2}\right)$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Use a double-integral to show that the area of the region R shown is $\frac{9\pi}{2}$.

$$\text{Area of } R = \iint_R dA = \int \int r dr d\theta$$



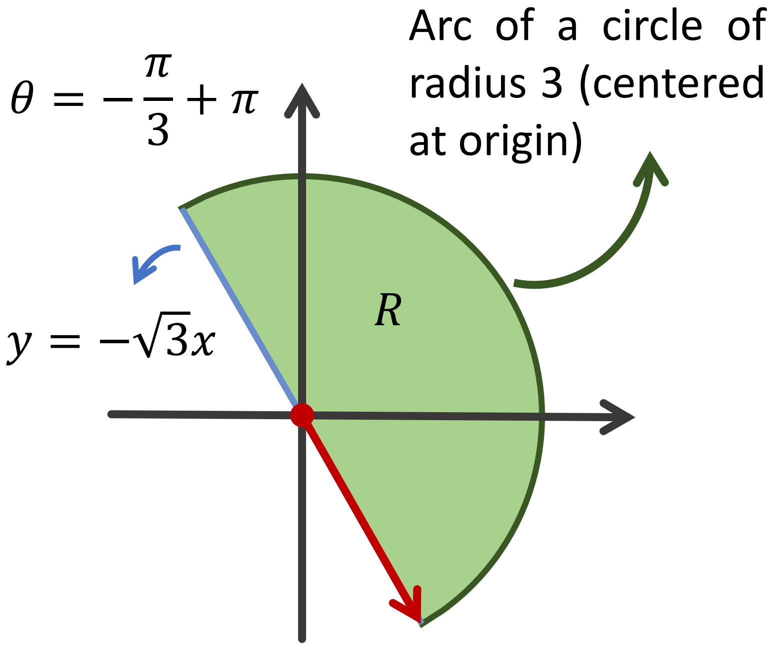
DOUBLE INTEGRALS IN POLAR COORDINATES

Example Use a double-integral to show that the area of the region R shown is $\frac{9\pi}{2}$.

$$\text{Area of } R = \iint_R dA = \int \int_0^3 r dr d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} \left[\int_0^3 r dr \right] d\theta = \int_{-\pi/3}^{2\pi/3} \frac{r^2}{2} \Big]_0^3 d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} \frac{9}{2} d\theta = \frac{9}{2}\theta \Big]_{-\pi/3}^{2\pi/3} = \frac{9\pi}{2}$$



$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}x}{x} = -\sqrt{3}$$

$$\theta = -\frac{\pi}{3}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\int_0^\infty e^{-x^2} dx = I$

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-x^2} dx\right)$$

$$= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dxdy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

DOUBLE INTEGRALS IN POLAR COORDINATES

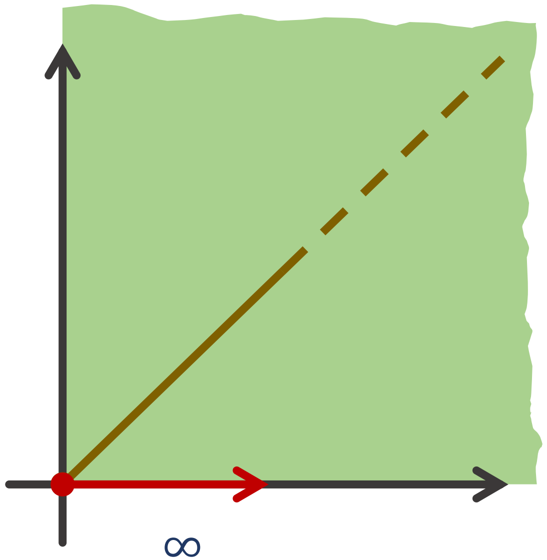
Example Evaluate $\displaystyle\int_0^\infty e^{-x^2}dx = I$

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int \int e^{-r^2} r dr d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^\infty r\, e^{-r^2} dr \right] d\theta$$

By substitution. Let $t = r^2$.

$$= \int_0^{\pi/2} \left[\int_0^\infty \frac{1}{2} e^{-t} dt \right] d\theta = \int_0^{\pi/2} \left[\frac{-1}{2} e^{-t} \right]_0^\infty d\theta = \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}$$



$$\therefore I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Course: Calculus (4)

Chapter: [14]

MULTIPLE INTEGRALS

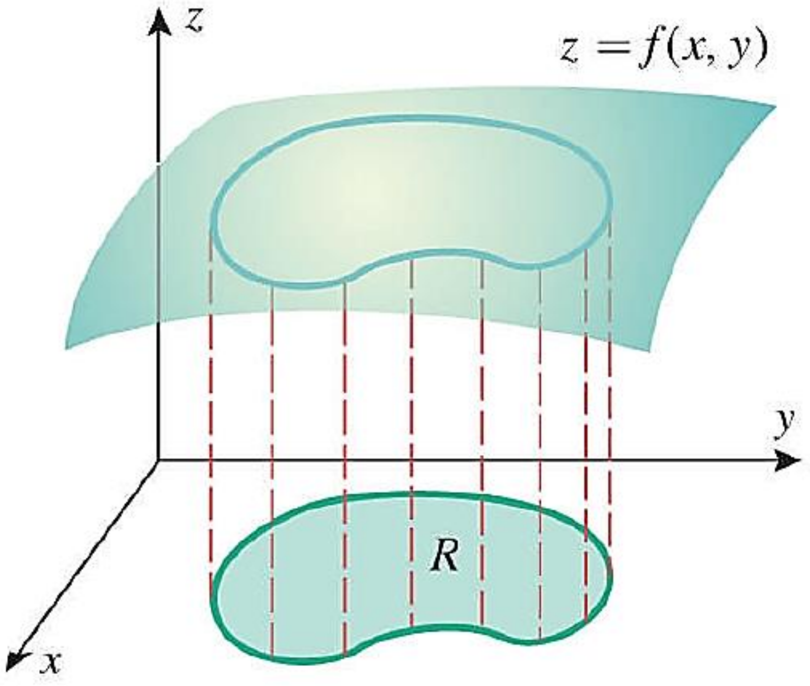
Section: [14.4]

SURFACE AREA; PARAMETRIC SURFACES

SURFACE AREA FOR SURFACES OF THE FORM $z = f(x, y)$

- Consider a surface of the form $z = f(x,y)$ defined over a region R in the xy −plane.
- We will assume that f has continuous first partial derivatives at the interior points of R.
- The surface area of that portion of the surface $z = f(x,y)$ that lies above the rectangle R in the xy −plane is given by

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

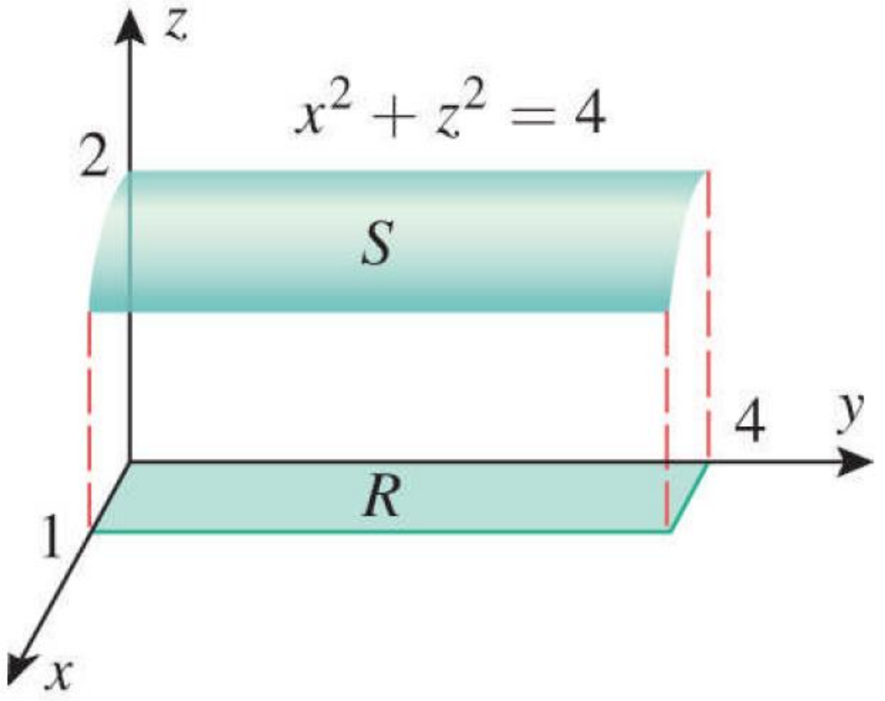


SURFACE AREA FOR SURFACES OF THE FORM $z = f(x, y)$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Example

Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy −plane whose coordinates satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 4$.



$$S = \int_0^1 \int_0^4 \sqrt{\left(\frac{-2x}{2\sqrt{4 - x^2}}\right)^2 + 0^2 + 1}\, dydx$$

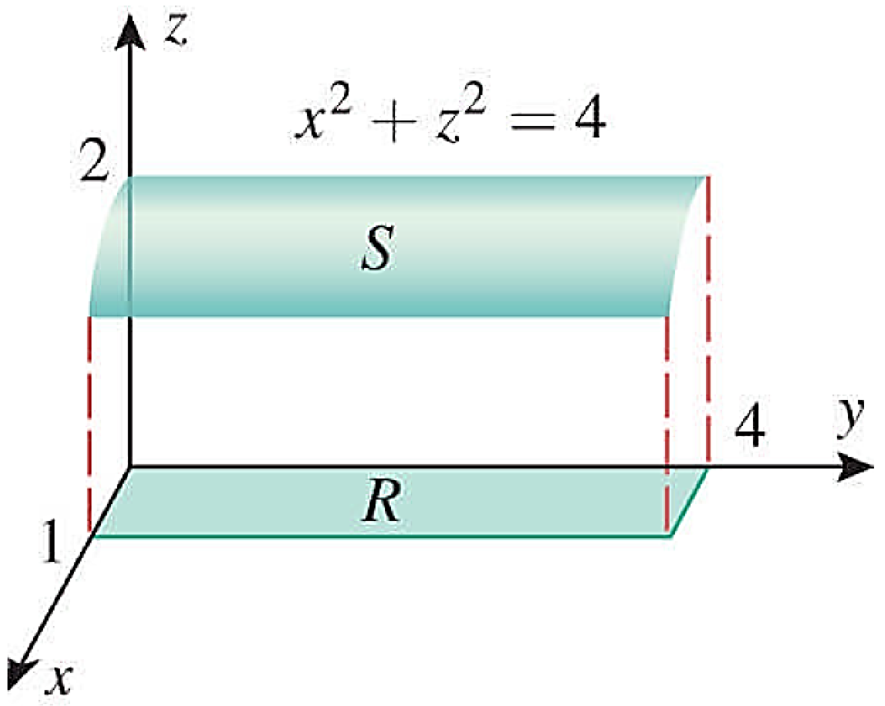
$$= \int_0^1 \int_0^4 \sqrt{\frac{x^2}{4 - x^2} + 1}\, dydx = \int_0^1 \int_0^4 \sqrt{\frac{4}{4 - x^2}}\, dydx$$

SURFACE AREA FOR SURFACES OF THE FORM $z = f(x, y)$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Example

Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy −plane whose coordinates satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 4$.



$$S = \int_0^1 \int_0^4 \frac{2}{\sqrt{4 - x^2}} dy dx = \int_0^1 \frac{8}{\sqrt{4 - x^2}} dx$$

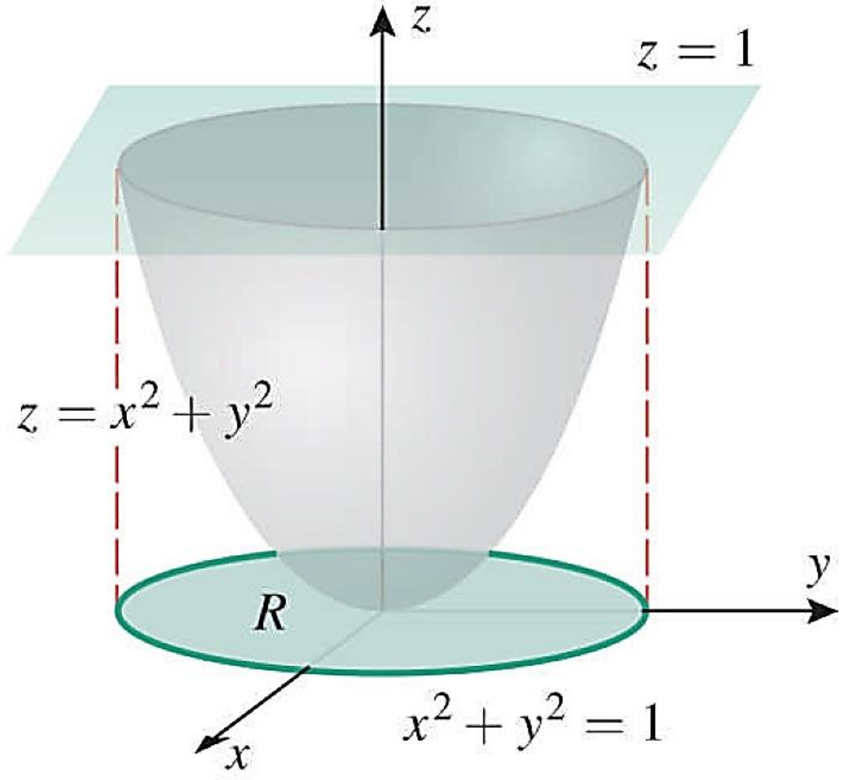
$$= 8 \sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 = 8\left(\frac{\pi}{6} - 0\right) = \frac{4\pi}{3}$$

SURFACE AREA FOR SURFACES OF THE FORM $z = f(x, y)$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Example

Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.



$$S = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1}\, dA$$

$$= \iint_R \sqrt{4(x^2 + y^2) + 1}\ dA$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r dr d\theta$$

By Substitution:

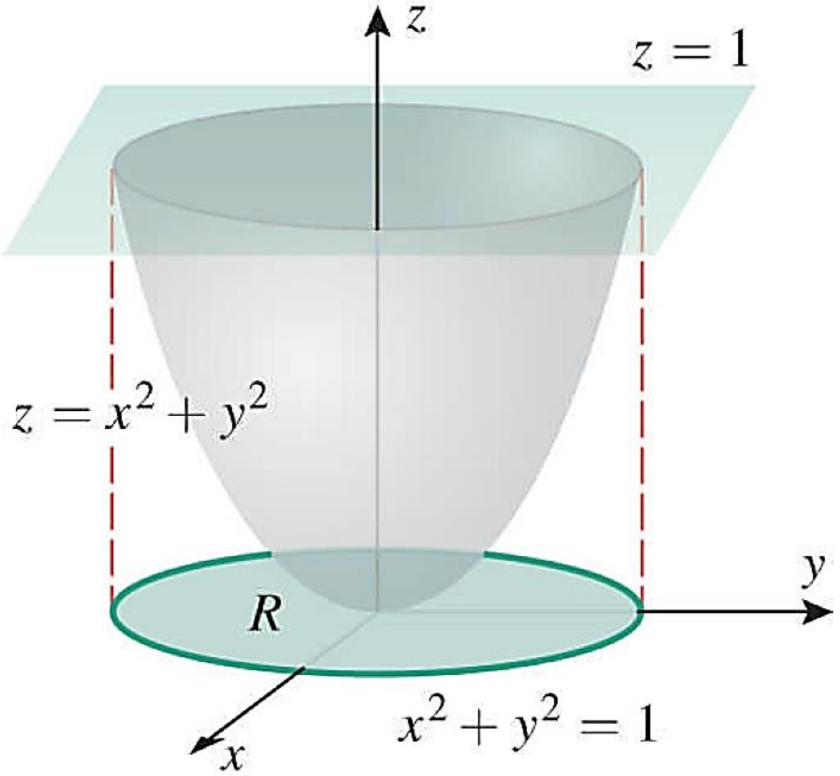
Let $t = 4r^2 + 1$

SURFACE AREA FOR SURFACES OF THE FORM $z = f(x,y)$

Example

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.



$$S = \int_0^{2\pi} \left[\int_1^5 \frac{1}{8}\sqrt{t}\, dt \right] d\theta = \int_0^{2\pi} \frac{1}{12}\sqrt{t^3}\Big]_1^5\, d\theta$$

$$= \int_0^{2\pi} \frac{5\sqrt{5}-1}{12}\, d\theta = \frac{1}{6}(5\sqrt{5}-1)\pi$$

PARAMETRIC REPRESENTATION OF SURFACES

We have seen that curves in 2-space can be represented by two equations involving one parameter, say

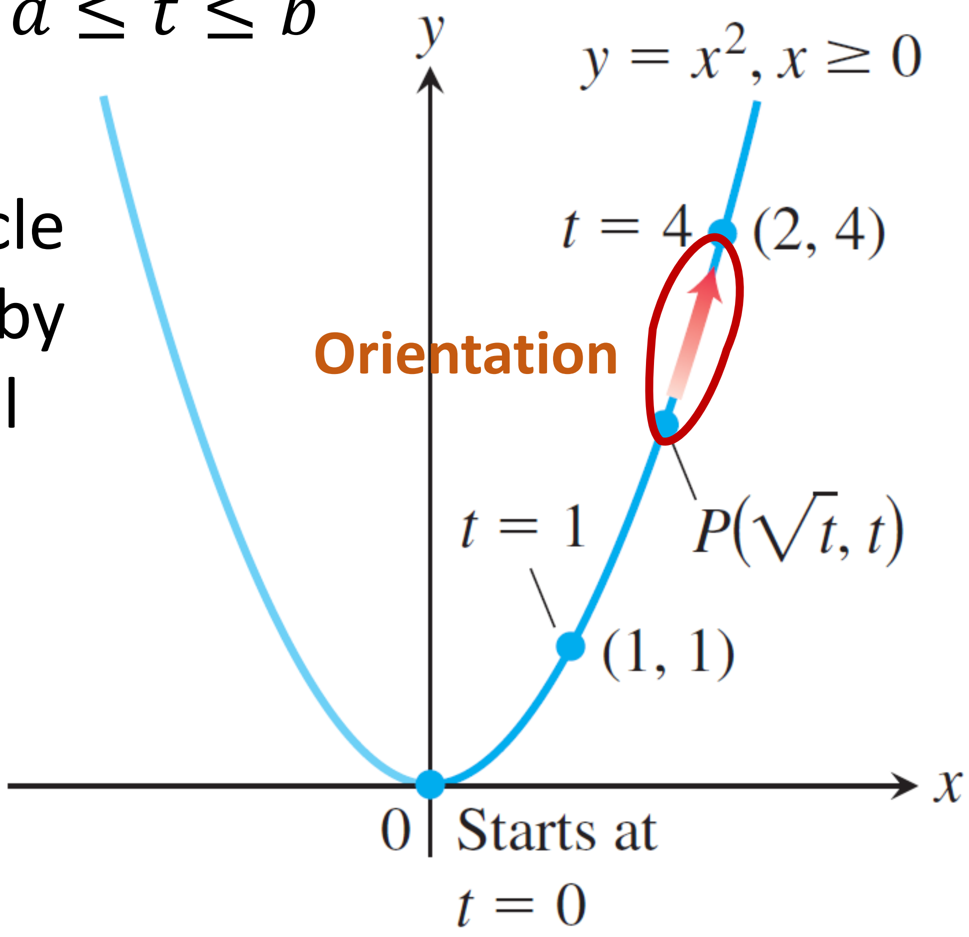
$$x = x(t) \quad , \quad y = y(t) \quad , \quad a \le t \le b$$

Example The position $P(x, y)$ of a particle moving in the xy −plane is given by the equations and parameter interval

$$x = \sqrt{t} \quad , \quad y = t \quad , \quad t \ge 0$$

We try to *identify the path* by eliminating t between the equations:

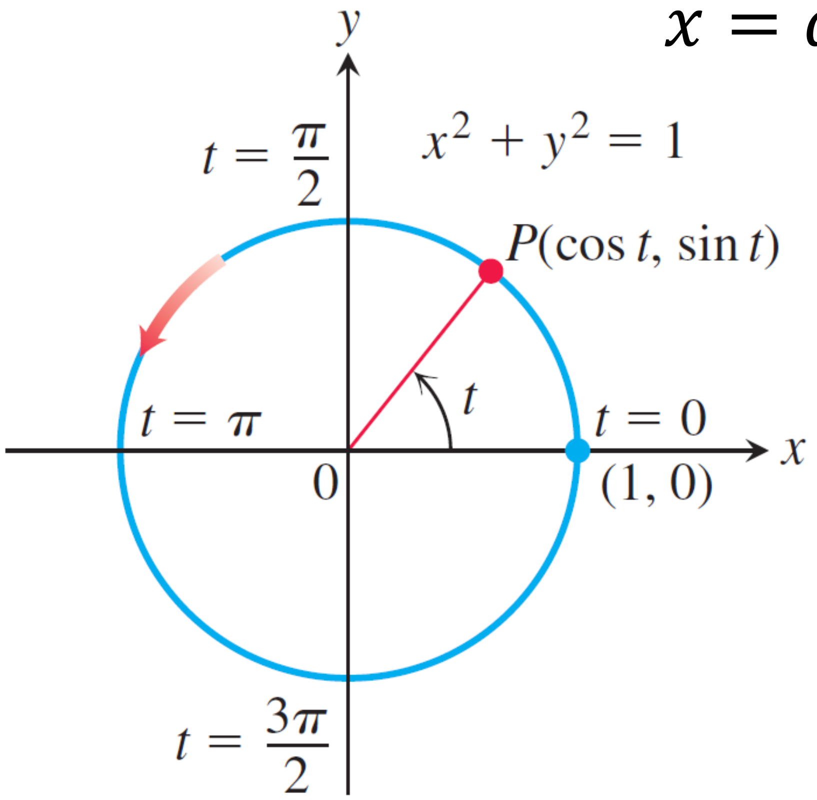
$$y = t = \left(\sqrt{t}\right)^2 = x^2$$



PARAMETRIC REPRESENTATION OF SURFACES

Example The *counter-clockwise* orientation parametric equations of the circle $x^2 + y^2 = a^2$ are

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad 0 \le t \le 2\pi$$



$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2.$$

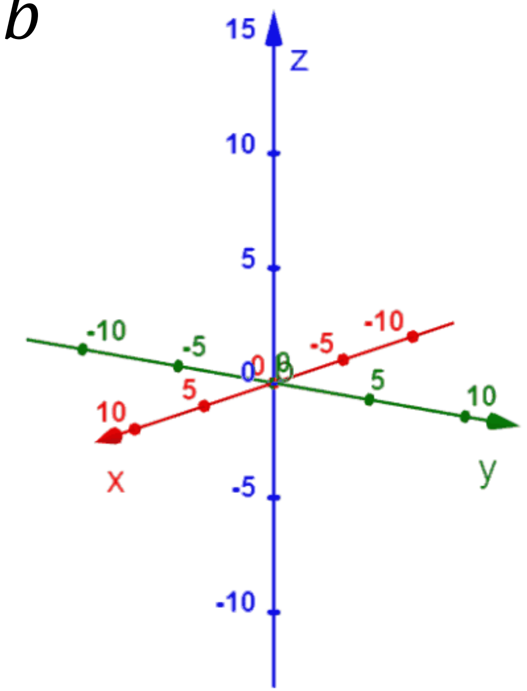
GeoGebra:

```
Curve(3cos(t), 3sin(t), t, 0, 2pi)
```

PARAMETRIC REPRESENTATION OF SURFACES

Curves in 3-space can be represented by three equations involving one parameter, say

$$x = x(t) \quad , \quad y = y(t) \quad , \quad z = z(t) \quad , \quad a \leq t \leq b$$



Example Describe the parametric curve represented by the equations

$$x = 10\cos t$$
$$y = 10\sin t$$
$$z = t$$

GeoGebra:

```
Curve(10cos(t), 10sin(t), t, t, 0, 6π)
```

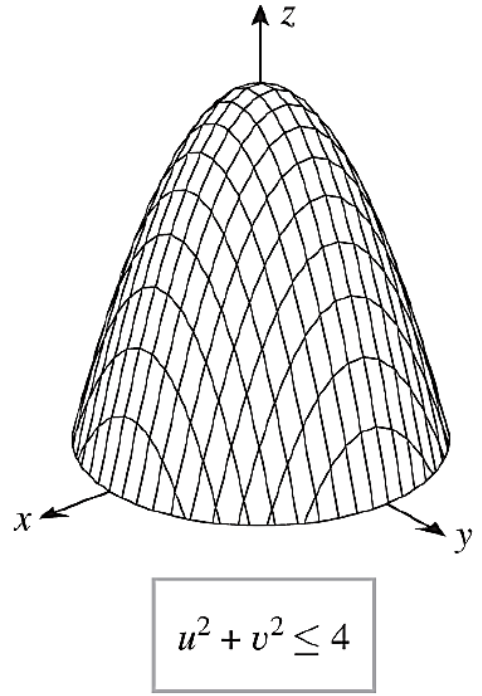
PARAMETRIC REPRESENTATION OF SURFACES

Surfaces in 3-space can be represented parametrically by three equations involving two parameters, say

$$x = x(u,v) \quad , \quad y = y(u,v) \quad , \quad z = z(u,v) \quad , \quad \begin{array}{c} a \le u \le b \\ c \le v \le d \end{array}$$

Example Consider the paraboloid $z = 4 - x^2 - y^2$. One way to parametrize this surface is to take

$$x = u$$
$$y = v$$
$$z = 4 - u^2 - v^2$$



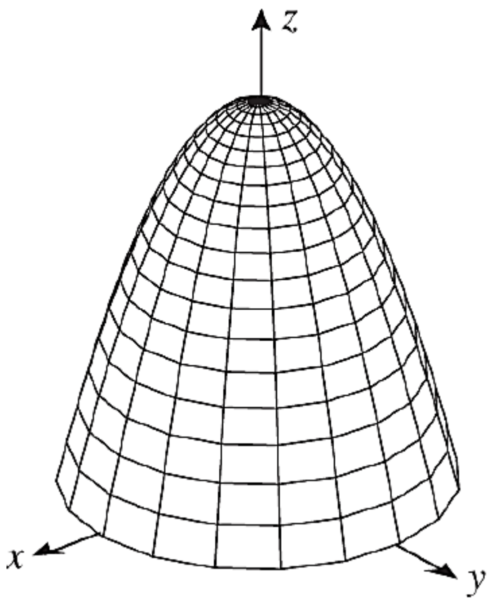
GeoGebra:

```
1) 4-x^2-y^2 , x^2+y^2<=4
2) Surface(u, v, 4-u²-v², u,-2,2, v,-2,2)
```

PARAMETRIC REPRESENTATION OF SURFACES

Example Consider the paraboloid $z = 4 - x^2 - y^2$. Another way to parametrize this surface is to take

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 4 - r^2$$



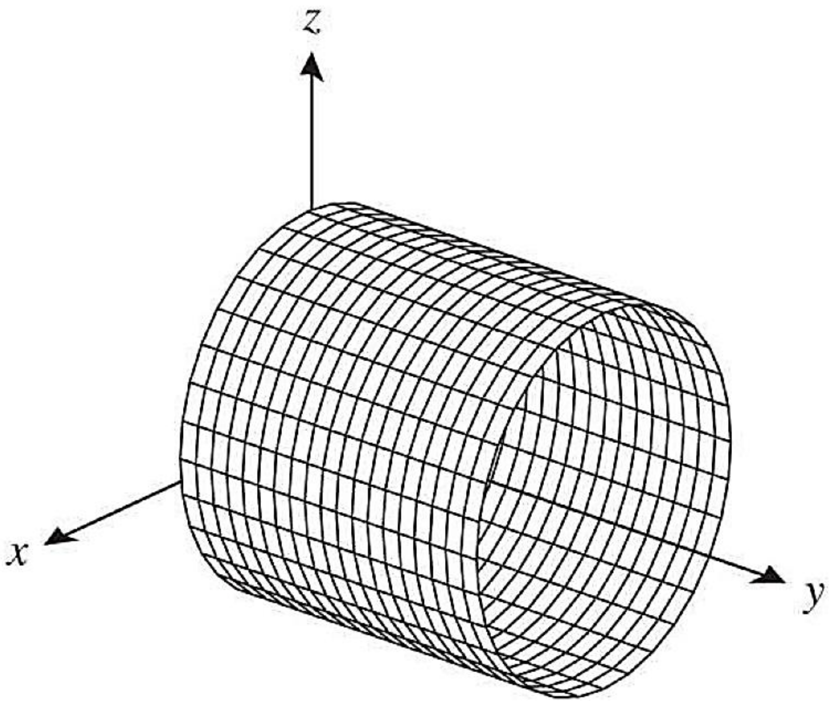
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

GeoGebra:

```
Surface(r cos(θ), r sin(θ), 4 - r², r, 0, 2, θ, 0, 2π)
```

PARAMETRIC REPRESENTATION OF SURFACES

Example Find parametric equations for the portion of the right circular cylinder $x^2 + z^2 = 9$ for which $0 \leq y \leq 5$ in terms of the parameters u and v.



$$x = 3\cos u$$

$$y = v$$

$$z = 3\sin u$$

GeoGebra:

```
Surface(3cos(u), v, 3sin(u), u,0,2π, v,0,5)
```

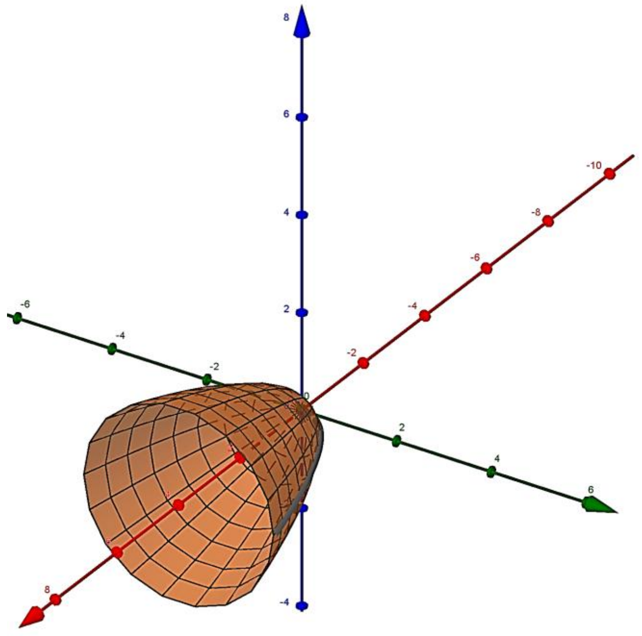
REPRESENTING SURFACES OF REVOLUTION PARAMETRICALLY

Suppose that we want to find parametric equations for the surface generated by revolving the plane curve $y = f(x)$ about the x −axis for example. Then the surface can be represented parametrically as

$$x = u \qquad y = f(u)\cos v \qquad z = f(u)\sin v$$

Example Find parametric equations for the surface generated by revolving the curve $y = \sqrt{x}$ about the x −axis.

$$x = u$$
$$y = \sqrt{u}\cos v \qquad 0 \le u \le 4$$
$$z = \sqrt{u}\sin v \qquad 0 \le v \le 2\pi$$



REPRESENTING SURFACES OF REVOLUTION PARAMETRICALLY

Example Find parametric equations for the surface generated by revolving the curve $y = \sqrt{u}$ about the x −axis.

$$x = u \qquad y = \sqrt{u}\cos v \qquad z = \sqrt{u}\sin v \qquad \begin{matrix} 0 \le u \le 4 \\ 0 \le v \le 2\pi \end{matrix}$$

GeoGebra:

Step [1] `f(x) = sqrt(x)`

Step [2] `Surface(u, f(u)cos(v), f(u)sin(v), u,0,4, v,0, 2π)`

GeoGebra:

Step [1] `f(x) = sqrt(x)`

Step [2] `Surface(f, 2π, xAxis)`

Course: Calculus (4)

Chapter: [14]
MULTIPLE INTEGRALS

Section: [14.5]
Triple Integrals

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Let G be the rectangular box defined by the inequalities

$$a \leq x \leq b \quad , \quad c \leq y \leq d \quad , \quad k \leq z \leq \ell$$

If f is continuous on the region G, then

$$\iiint\limits_G f(x,y,z)dV = \int_a^b \int_c^d \int_k^\ell f(x,y,z)dzdydx$$

Six orders of integration are possible for the iterated integral:

$$\begin{array}{lll} dx\,dy\,dz, & dy\,dz\,dx, & dz\,dx\,dy \\ dx\,dz\,dy, & dz\,dy\,dx, & dy\,dx\,dz \end{array}$$

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Example Evaluate the triple integral $\iiint\limits_G 12xy^2z^3 dV$ over the rectangular box $G = [-1,2] \times [0,3] \times [0,2]$

$$\iiint\limits_G 12xy^2z^3 dV = \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3 dzdydx = \int_{-1}^{2}\int_{0}^{3}\left[\int_{0}^{2} 12xy^2z^3 dz\right] dydx$$

$$= \int_{-1}^{2}\int_{0}^{3} 48xy^2 dydx = \int_{-1}^{2} 432x dx = 648$$

$$\iiint\limits_G 12xy^2z^3 dV = 12\left[\int_{-1}^{2} x dx\right]\left[\int_{0}^{3} y^2 dy\right]\left[\int_{0}^{2} z^3 dz\right] = 648$$

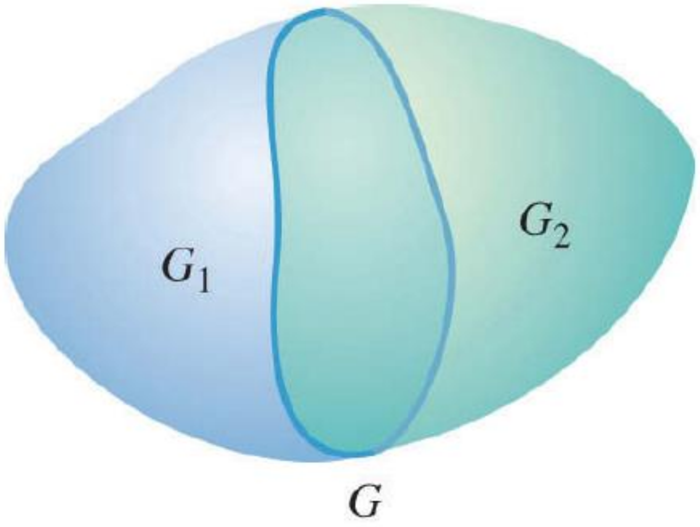
PROPERTIES OF TRIPLE INTEGRALS

$$\iiint\limits_G cf(x,y,z)dV = c\iiint\limits_G f(x,y,z)dV \quad \text{where } c \text{ is a constant.}$$

$$\iiint\limits_G (f \pm g)dV = \iiint\limits_G f dV \pm \iiint\limits_G g dV$$

If the region G is subdivided into two subregions G_1 and G_2, then

$$\iiint\limits_G f dV = \iiint\limits_{G_1} f dV + \iiint\limits_{G_2} f dV$$



EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Example Evaluate $\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z \, dzdxdy$

$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z \, dzdxdy = \int_0^1 \int_0^y \frac{1}{2} z^2 \Big]_0^{\sqrt{1-y^2}} dxdy = \int_0^1 \int_0^y \frac{1}{2}(1-y^2)dxdy$$

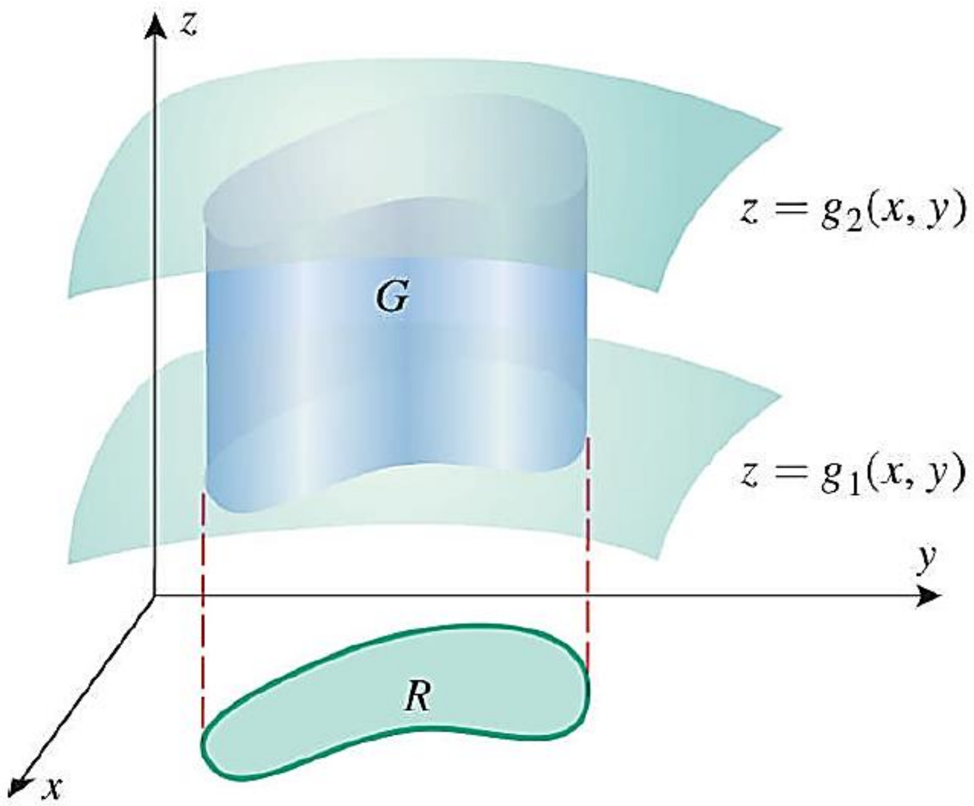
$$= \int_0^1 \frac{1}{2}(1-y^2)x\Big]_0^y dy = \int_0^1 \frac{1}{2}(1-y^2)ydy$$

$$= \frac{1}{2}\int_0^1 (y-y^3)dy = \frac{1}{8}$$

EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Let G be a simple xy −solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy −plane. If $f(x, y, z)$ is continuous on G, then

$$\iiint\limits_G f(x,y,z)dV = \iint\limits_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz \right] dA$$



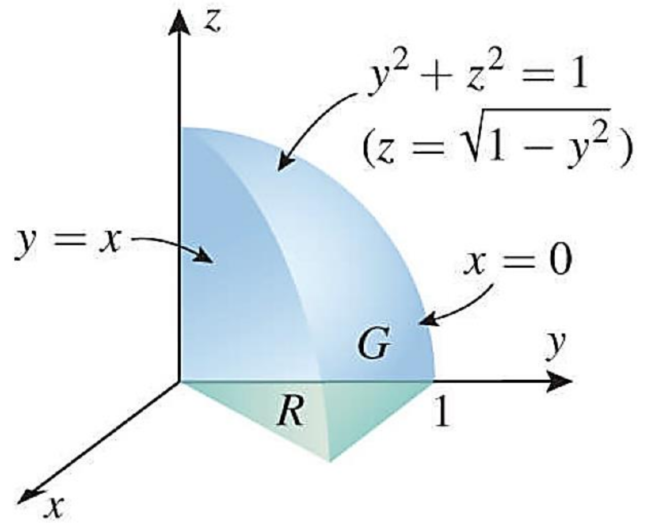
EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

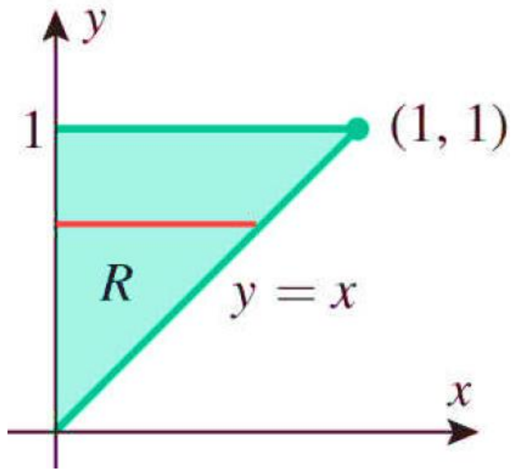
Example Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\, dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\, dz\right] dA$$

$$= \iint_R \left[\frac{1}{2}z^2\right]_0^{\sqrt{1-y^2}} dA = \iint_R \frac{1}{2}(1-y^2)\, dA$$

$$= \int_0^1 \int_0^y \frac{1}{2}(1-y^2)\, dx\, dy = \frac{1}{8}$$

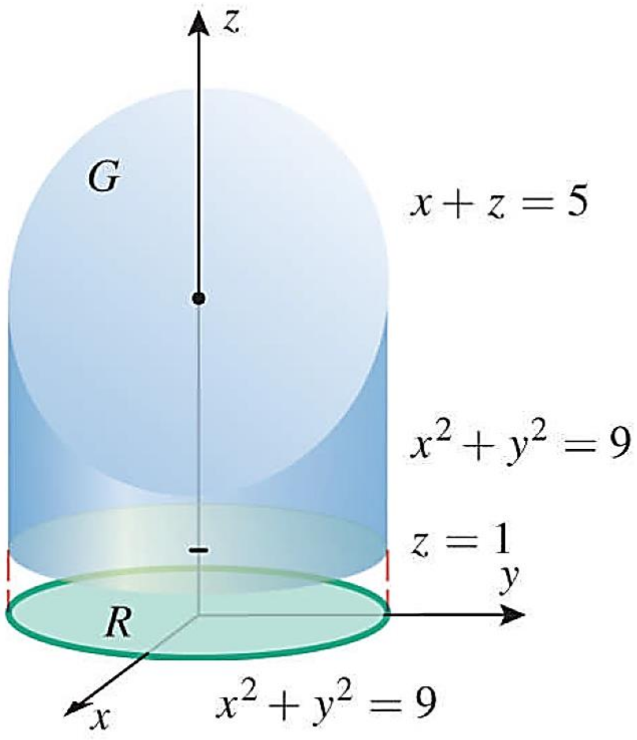




VOLUME CALCULATED AS A TRIPLE INTEGRAL

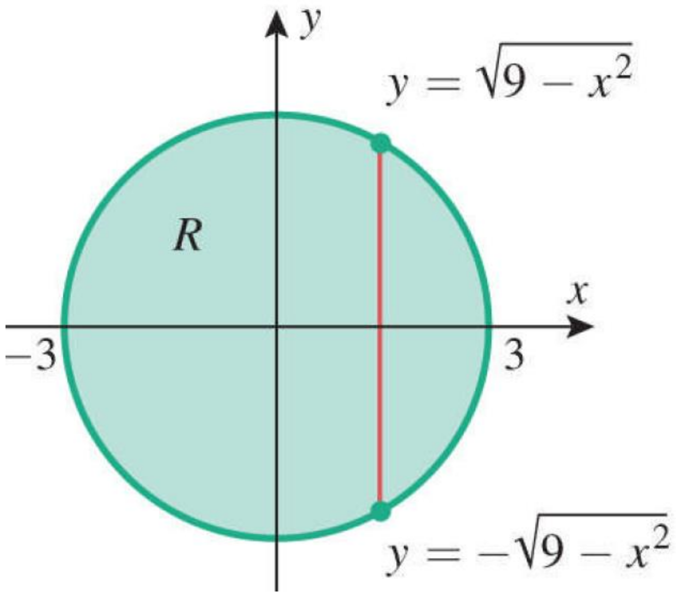
NOTE $\quad \text{Volume of } G = \iiint\limits_G dV$

Example Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.



$$\text{Volume of } G = \iiint\limits_G dV = \iint\limits_R \left[\int_1^{5-x} dz\right] dA = \iint\limits_R (4 - x)\, dA$$

$$= \int_0^{2\pi}\int_0^3 (4 - r\cos\theta)\, r\, dr\, d\theta = 36\pi$$



Cylindrical Coordinates

Course: Calculus (4)

Chapter: [14]

MULTIPLE INTEGRALS

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TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Course: Calculus (4)

Chapter: [14]

MULTIPLE INTEGRALS

Section: [14.7]

CHANGE OF VARIABLES IN MULTIPLE INTEGRALS; JACOBIANS

CHANGE OF VARIABLE IN A SINGLE INTEGRAL

- In many instances it is convenient to make a **substitution**, or **change of variable**, in an integral to evaluate it.
- If f is continuous and $x = g(u)$ has a continuous derivative and $dx = g'(u)du$, then

$$\int_a^b f(x)dx = \int_c^d f\big(g(u)\big)g'(u)du = \int_c^d f\big(g(u)\big)\,J(u)du$$

- **For example**, to evaluate $\int_0^2 \sqrt{4-x^2}dx$ we use the substitution $x = 2\sin\theta$.

$$\int_0^2 \sqrt{4-x^2}dx \ = \int_0^{\pi/2} (2\cos\theta)(2\cos\theta)d\theta \ = 4\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi$$

CHANGE OF VARIABLE IN A SINGLE INTEGRAL

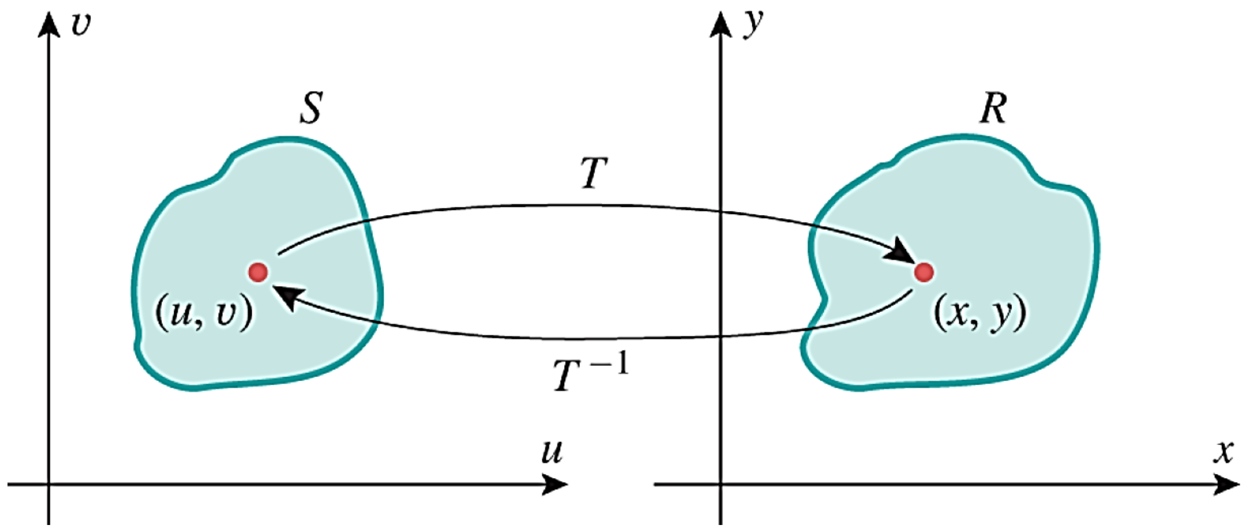
- In this section we will discuss a general method for evaluating double integrals by **substitution**.
- The **polar coordinate substitution** is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.
- We will consider parametric equations of the form

$$x = x(u, v) \quad , \quad y = (u, v)$$

- Parametric equations of this type associate points in the xy −plane with points in the uv −plane.

TRANSFORMATIONS OF THE PLANE

- If we think of the pair of numbers (u, v) as an input, then the two equations, in combination, produce a unique output (x, y), and hence define a function T that associates points in the xy −plane with points in the uv −plane.
- This function is described by the formula $T(u, v) = \big(x(u, v), y(u, v)\big)$.
- We call T a **transformation** from the uv −plane to the xy −plane.



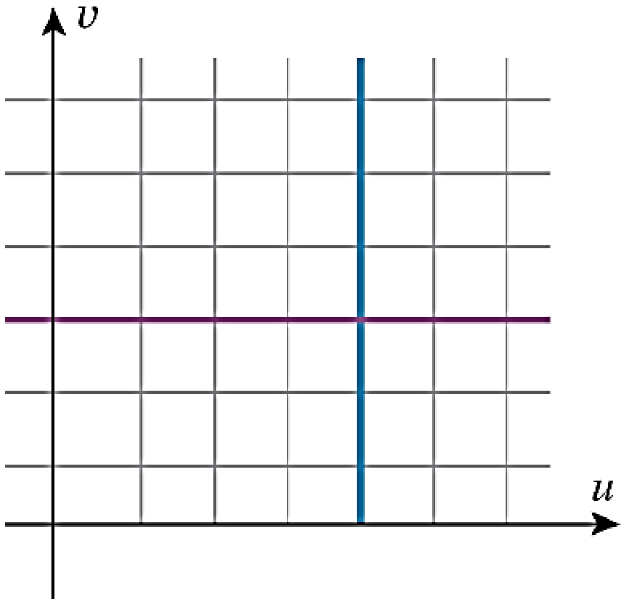
TRANSFORMATIONS OF THE PLANE

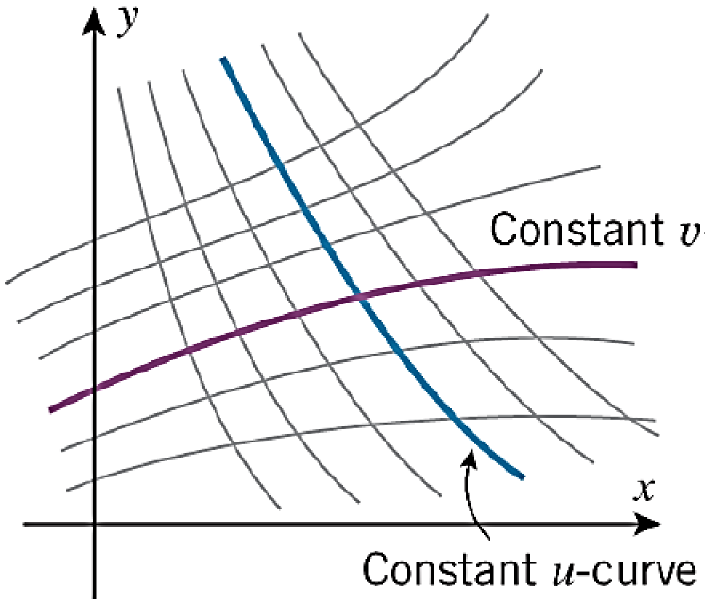
- These equations, which can often be obtained by solving for u and v in terms of x and y, define a transformation from the xy −plane to the uv −plane that maps the image of (u, v) under T back into (u, v). This transformation is denoted by T^{-1} and is called **the inverse of** T.

Because there are four variables involved, a three-dimensional figure is not very useful for describing the transformation geometrically. The idea here is to use the two planes to get the four dimensions needed.

TRANSFORMATIONS OF THE PLANE

- One way to visualize the geometric effect of a transformation T is to determine the images in the xy −plane of the vertical and horizontal lines in the uv −plane.
- Sets of points in the xy −plane that are images of horizontal lines (v constant) are called **constant $\boldsymbol{v}$ −curves**, and sets of points that are images of vertical lines (u constant) are called **constant $\boldsymbol{u}$ −curves**.





TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

a) Find $T(1,3)$.

Solution

Substituting $u = 1$ and $v = 3$ in the equations yields $T(1, 3) = (1, -1)$.

TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

b) Sketch the constant v −curves corresponding to $v = -2, -1, 0, 1, 2$.
c) Sketch the constant u −curves corresponding to $u = -2, -1, 0, 1, 2$.

Solution In these parts it will be convenient to express the transformation equations with u and v as functions of x and y.

$$\begin{aligned} 4x &= u + v \\ 2y &= u - v \quad + \\ \hline \end{aligned} \qquad\qquad \begin{aligned} 4x &= u + v \\ 2y &= u - v \quad - \\ \hline \end{aligned}$$

$$\boxed{u = 2x + y} \qquad\qquad \boxed{v = 2x - y}$$

TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

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c) Sketch the constant u −curves corresponding to $u = -2, -1, 0, 1, 2$.

Solution In these parts it will be convenient to express the transformation equations with u and v as functions of x and y.

$$u = 2x + y$$
$$v = 2x - y$$

The constant v −curves

$$-2 = 2x - y \qquad 0 = 2x - y$$
$$-1 = 2x - y \qquad 1 = 2x - y$$
$$2 = 2x - y$$

TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

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Solution In these parts it will be convenient to express the transformation equations with u and v as functions of x and y.

$$u = 2x + y$$

$$v = 2x - y$$

The constant u −curves

$$-2 = 2x + y \qquad 0 = 2x + y$$

$$-1 = 2x + y \qquad 1 = 2x + y$$

$$2 = 2x + y$$

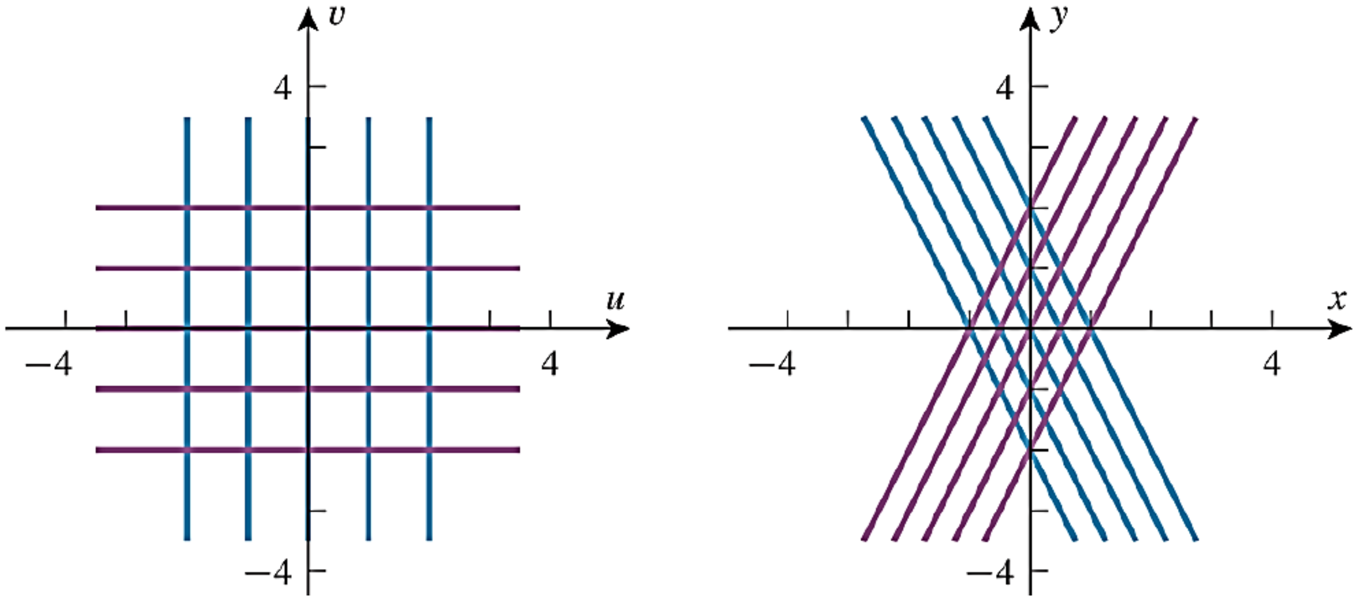
TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

b) Sketch the constant v −curves corresponding to $v = -2, -1, 0, 1, 2$.
c) Sketch the constant u −curves corresponding to $u = -2, -1, 0, 1, 2$.

Solution



$u = 2x + y$

$v = 2x - y$

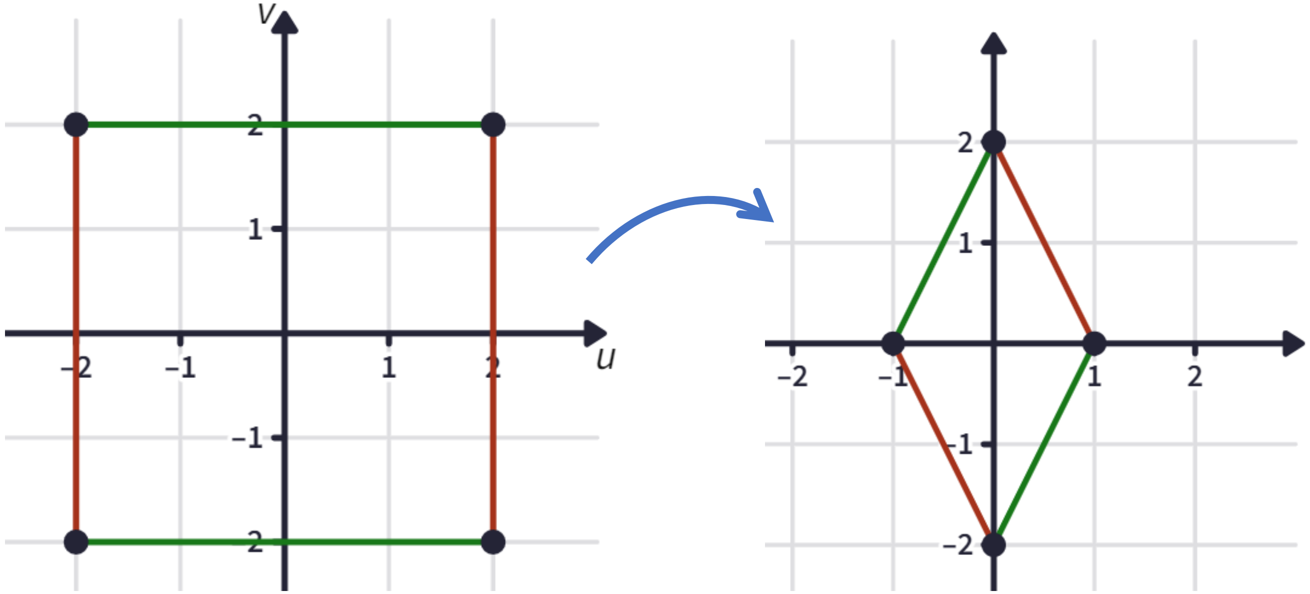
TRANSFORMATIONS OF THE PLANE

Example Let T be the transformation from the uv −plane to the xy −plane defined by the equations

$$x = \frac{1}{4}(u + v) \quad , \quad y = \frac{1}{2}(u - v)$$

d) Sketch the image under T of the square region in the uv −plane bounded by the lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$.

Solution



NOTE

Square Area $= 16$

Diamond Area $= 4$

$$dxdy = \frac{1}{4}\, dudv$$

JACOBIANS IN TWO VARIABLES

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

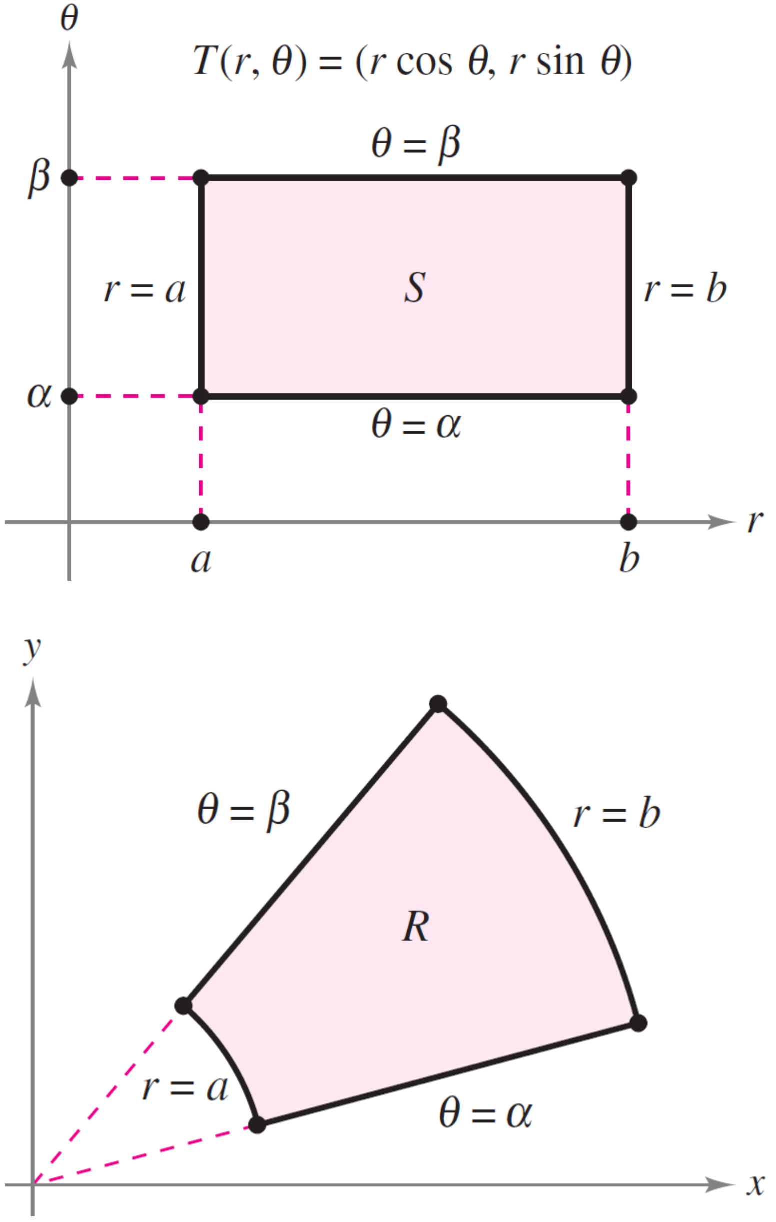
Example In the previous example, $x = \frac{1}{4}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then

$$J(u, v) = \begin{vmatrix} \dfrac{1}{4} & \dfrac{1}{4} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}$$

JACOBIANS IN TWO VARIABLES

Example Find the Jacobian for the change of variables defined by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$



$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\therefore\ dxdy = r\ drd\theta$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Let R be a simple region in the xy −plane and let S be a simple region in the uv −plane. Let T from S to R be given by

$$T(u, v) = \big(x(u, v)\, , y(u, v)\big)$$

where $x(u, v)$ and $y(u, v)$ have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R and $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero on S, then

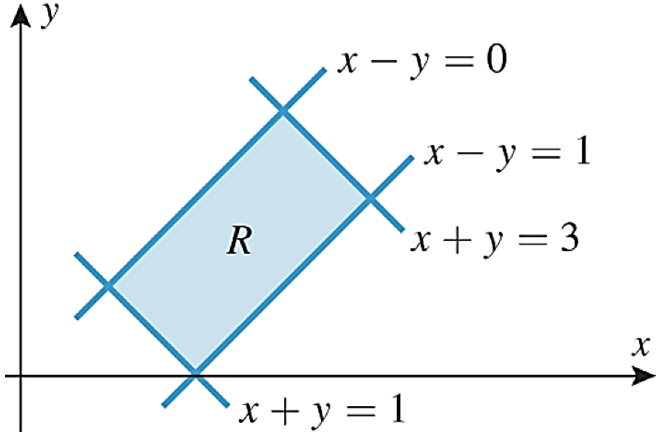
$$\iint\limits_R f(x, y) dA_{xy} = \iint\limits_S f\big(x(u, v), y(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA_{uv}$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example a) Let R be the region bounded by the lines

$$x - y = 0, \quad x - y = 1, \quad x + y = 1, \quad \text{and} \quad x + y = 3$$

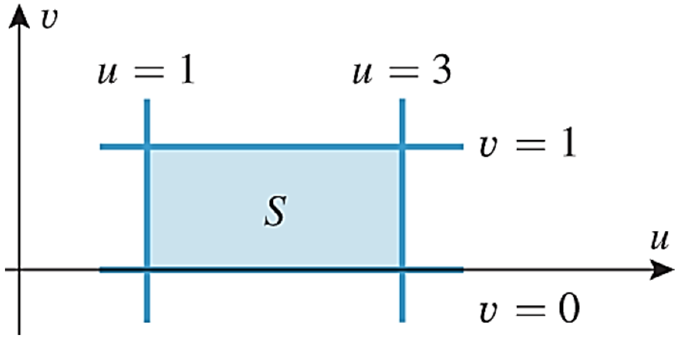
as shown in the figure. Find a transformation T from a region S to R such that is S a rectangular region in the uv −plane.



$$u = x - y \qquad 0 \leq u \leq 1$$
$$v = x + y \qquad 1 \leq v \leq 3$$

To find the transformation T:

$$\begin{array}{ll} u = x - y & \\ v = x + y & + \\ \hline x = \dfrac{1}{2}(v + u) & \end{array} \qquad \begin{array}{ll} u = x - y & \\ v = x + y & - \\ \hline y = \dfrac{1}{2}(v - u) & \end{array}$$



CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example b) Evaluate $\iint_R \frac{x-y}{x+y}\, dA$

$u = x - y$

$v = x + y$

$x = \frac{1}{2}(v + u)$

$y = \frac{1}{2}(v - u)$

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) = \frac{1}{2}$$

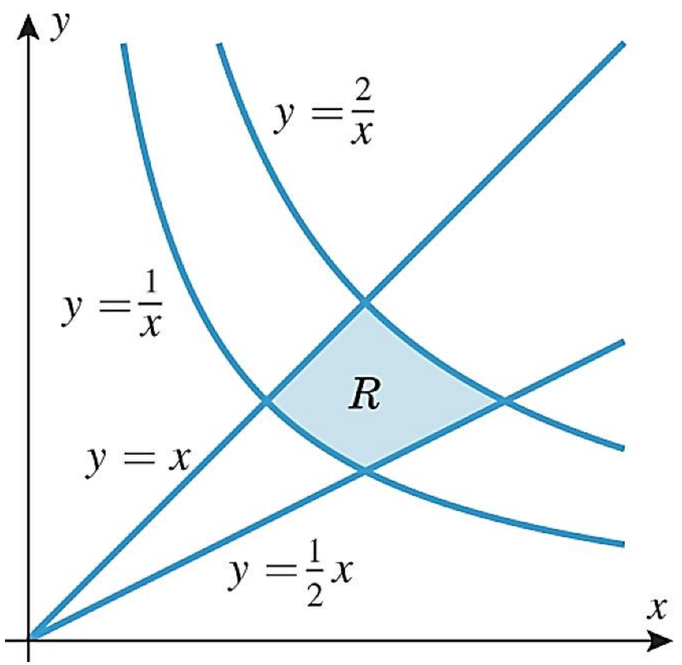
$$\iint\limits_R \frac{x-y}{x+y}\, dA = \iint\limits_S \frac{u}{v}\, |J(u,v)|\, dA_{uv} = \frac{1}{2}\int_1^3 \int_0^1 \frac{u}{v}\, du dv$$

$$= \frac{1}{4}\int_1^3 \frac{1}{v} dv = \frac{1}{4}\ln 3$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

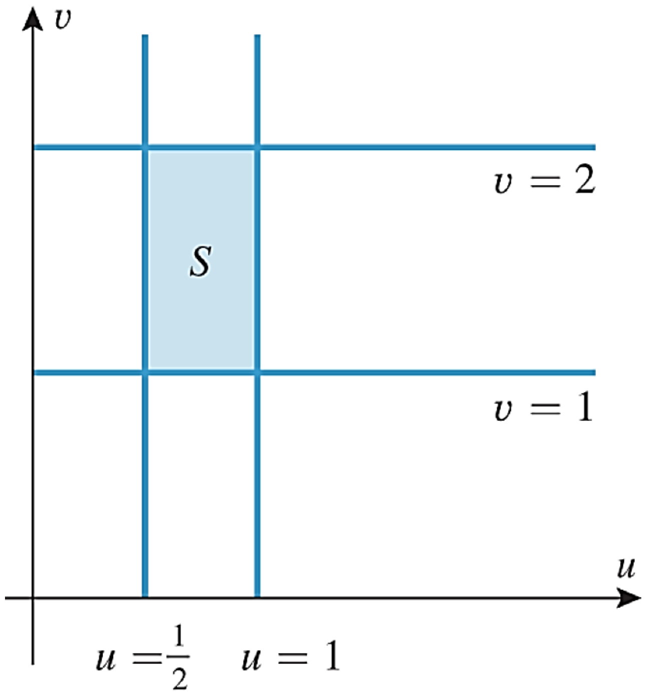
Example Let R be the region enclosed by the lines $y = \frac{1}{2}x$ and $y = x$, and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$. Evaluate

$$\iint_R e^{xy}\, dA$$



$$\left.\begin{array}{l} \dfrac{y}{x} = \dfrac{1}{2} \\ \dfrac{y}{x} = 1 \end{array}\right\} \quad u = \frac{y}{x} \qquad \frac{1}{2} \le u \le 1$$

$$\left.\begin{array}{l} xy = 1 \\ xy = 2 \end{array}\right\} \quad v = xy \qquad 1 \le v \le 2$$



CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example Let R be the region enclosed by the lines $y = \frac{1}{2}x$ and $y = x$, and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$. Evaluate

$$\iint_R e^{xy}\, dA$$

$$\boxed{u = \frac{y}{x}} \quad \frac{1}{2} \le u \le 1$$

$$\boxed{v = xy} \quad 1 \le v \le 2$$

$$u\,v = \frac{y}{x} \cdot xy = y^2 \;\Rightarrow\; y = \sqrt{uv}$$

$$\frac{u}{v} = \frac{y/x}{xy} = 1/x^2 \;\Rightarrow\; x = \sqrt{\frac{v}{u}}$$

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}\sqrt{\frac{v}{u^3}} & \frac{1}{2}\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= -\frac{1}{4}\frac{1}{u} - \frac{1}{4}\frac{1}{u} = -\frac{1}{2}\frac{1}{u}$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example Let R be the region enclosed by the lines $y = \frac{1}{2}x$ and $y = x$, and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$. Evaluate

$$\iint_R e^{xy}\, dA$$

$$\iint_R e^{xy}\, dA = \iint_S e^{v}\, |J(u,v)|\, dA_{uv} = \frac{1}{2}\int_1^2 \int_{1/2}^1 \frac{1}{u} e^{v} du dv$$

$$= \frac{1}{2}\left[\int_1^2 e^{v} dv\right]\left[\int_{1/2}^1 \frac{1}{u} du\right] = \frac{1}{2} e(e-1)\ln 2$$

$u = \frac{y}{x}$ $\quad \frac{1}{2} \le u \le 1$

$v = xy$ $\quad 1 \le v \le 2$

$y = \sqrt{uv}$

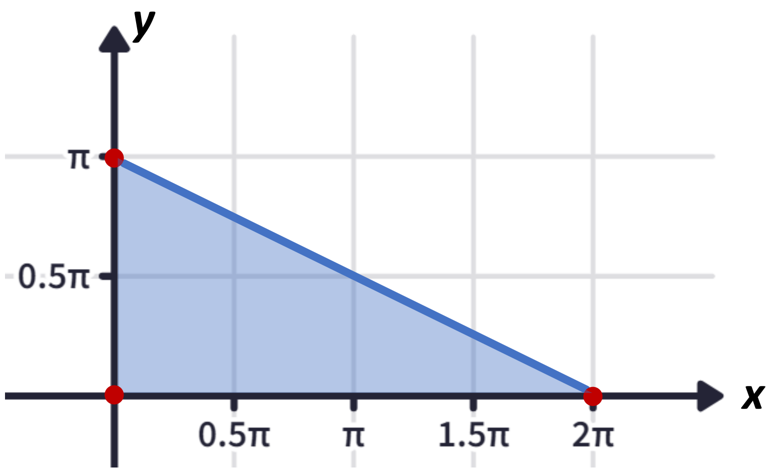
$x = \sqrt{\frac{v}{u}}$

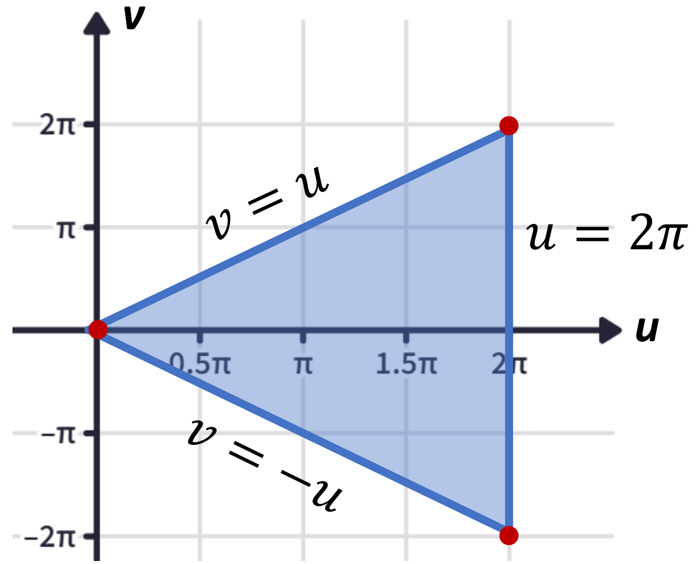
$J(u,v) = -\frac{1}{2}\frac{1}{u}$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example Let R be the region bounded by the line $x + 2y = 2\pi$, $y -$ axis, and $x -$ axis. Evaluate

$$\iint_R \sin(x + 2y)\cos(x - 2y)\ dA$$





Since it is not easy to integrate $\sin(x + 2y)\cos(x - 2y)$, we make a change of variables suggested by:

$$\begin{array}{l} u = x + 2y \\ v = x - 2y \end{array} \quad +$$

$$x = \frac{1}{2}(u + v)$$

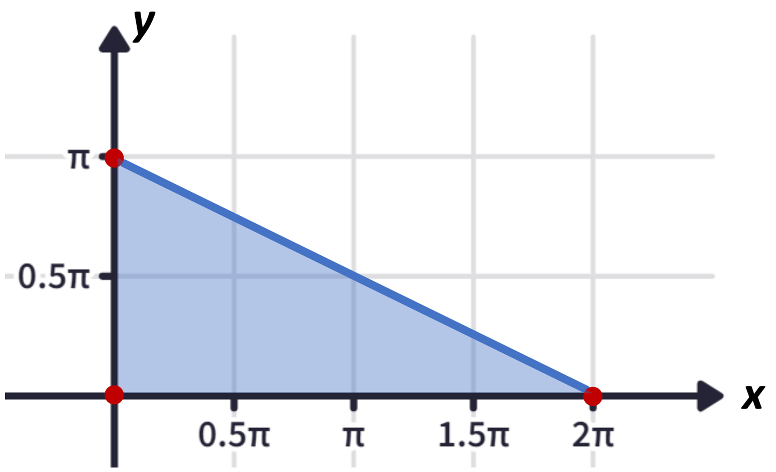
$$\begin{array}{l} u = x + 2y \\ v = x - 2y \end{array} \quad -$$

$$y = \frac{1}{4}(u - v)$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example Let R be the region bounded by the line $x + 2y = 2\pi$, $y -$ axis, and $x -$ axis. Evaluate

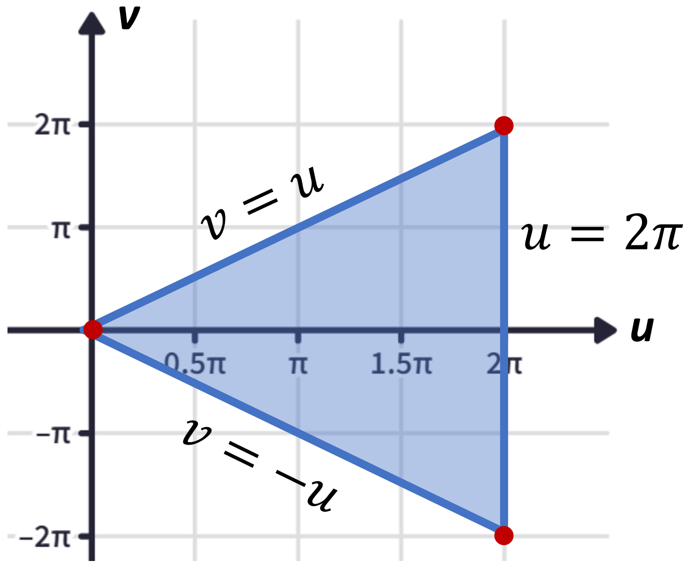
$$\iint_R \sin(x+2y)\cos(x-2y)\ dA$$



$$x = \frac{1}{2}(u+v)$$

$$y = \frac{1}{4}(u-v)$$

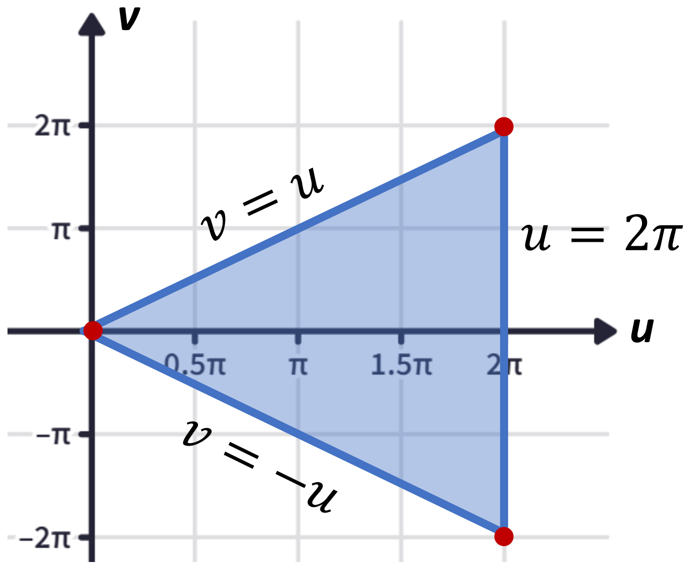
$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}$$



CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Example Let R be the region bounded by the line $x + 2y = 2\pi$, $y -$ axis, and $x -$ axis. Evaluate

$$\iint_R \sin(x+2y)\cos(x-2y)\ dA$$



Type I Region

$$= \iint_S \sin(u)\cos(v)\,|J(u,v)|\ dA_{uv}$$

$$= \frac{1}{4}\int_0^{2\pi}\int_{-u}^{u} \sin(u)\cos(v)\,dvdu = \frac{1}{4}\int_0^{2\pi} [\sin(u)\sin(v)]_{-u}^{u}\,du = \frac{1}{2}\int_0^{2\pi} \sin^2(u)\,du = \frac{\pi}{2}$$

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MULTIPLE INTEGRALS

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CENTERS OF GRAVITY USING MULTIPLE INTEGRALS