Course: Calculus (4)

Chapter: [15]

TOPICS IN VECTOR CALCULUS

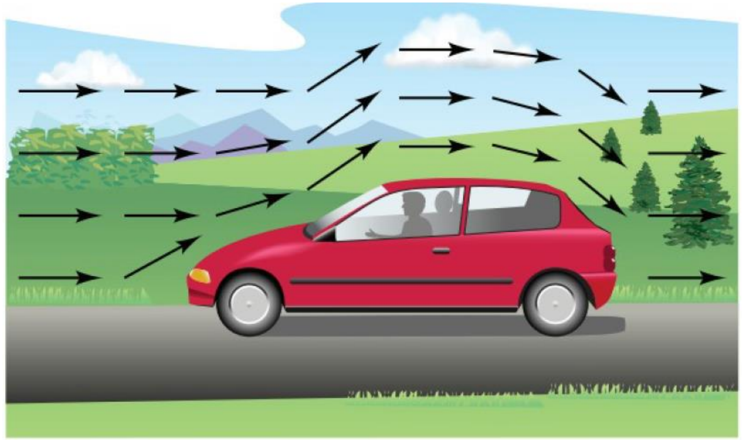
Section: [15.1]

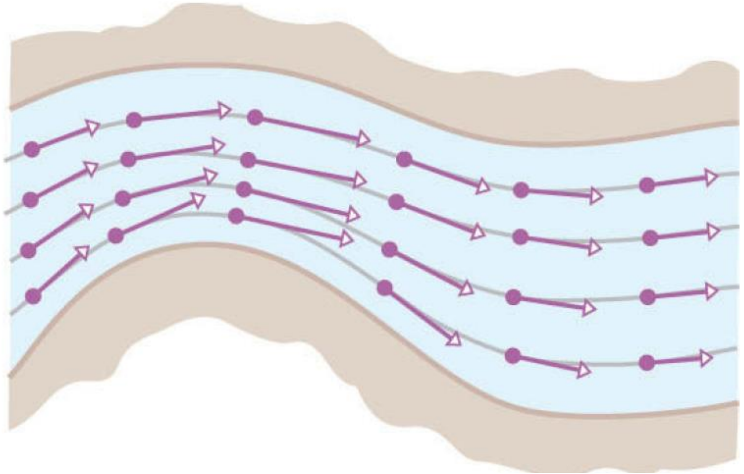
VECTOR FIELDS

VECTOR FIELDS

- A **vector field** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane.

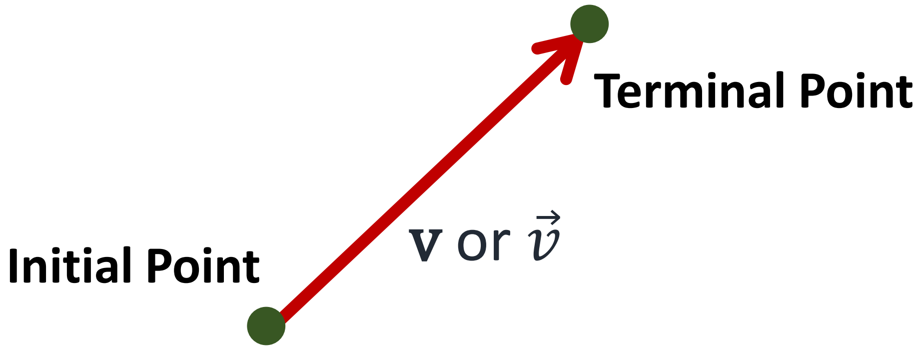
$$\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$$



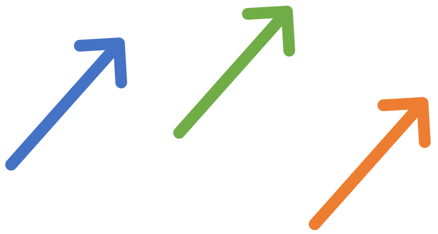


VECTORS VIEWED GEOMETRICALLY

A **Vector** in 2-space or 3-space; is an **arrow** with direction and length (magnitude).

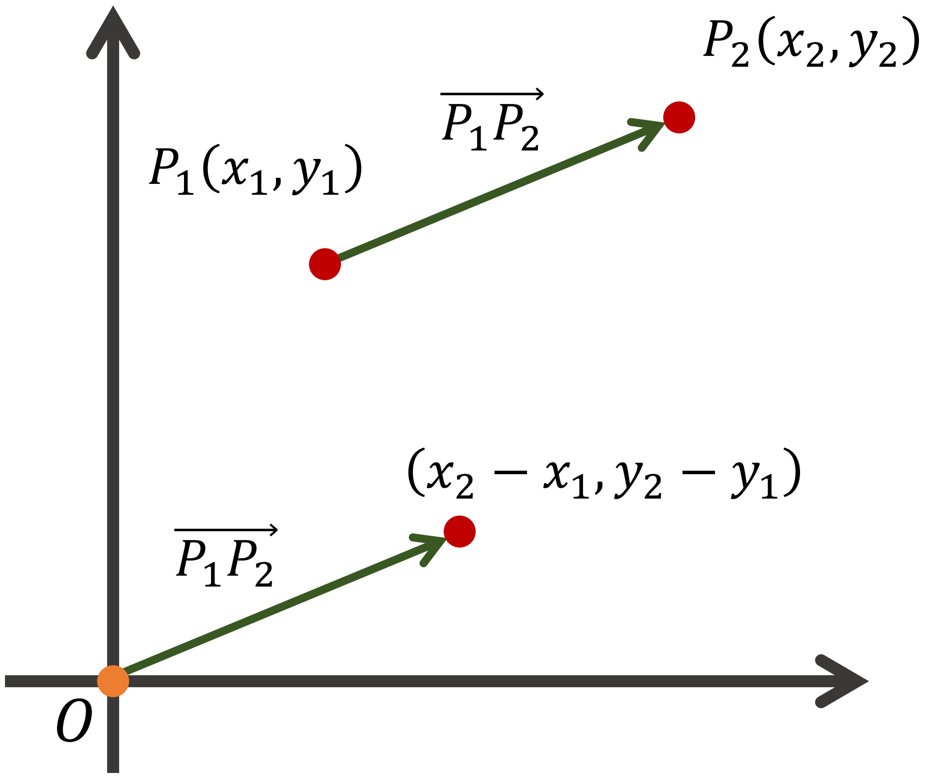


Two vectors are equal if they are translations of one another.



VECTORS VIEWED GEOMETRICALLY

Because vectors are **not** *affected* by translation, the initial point of a vector **v** can be *moved* to any convenient point A by making an appropriate *translation*.

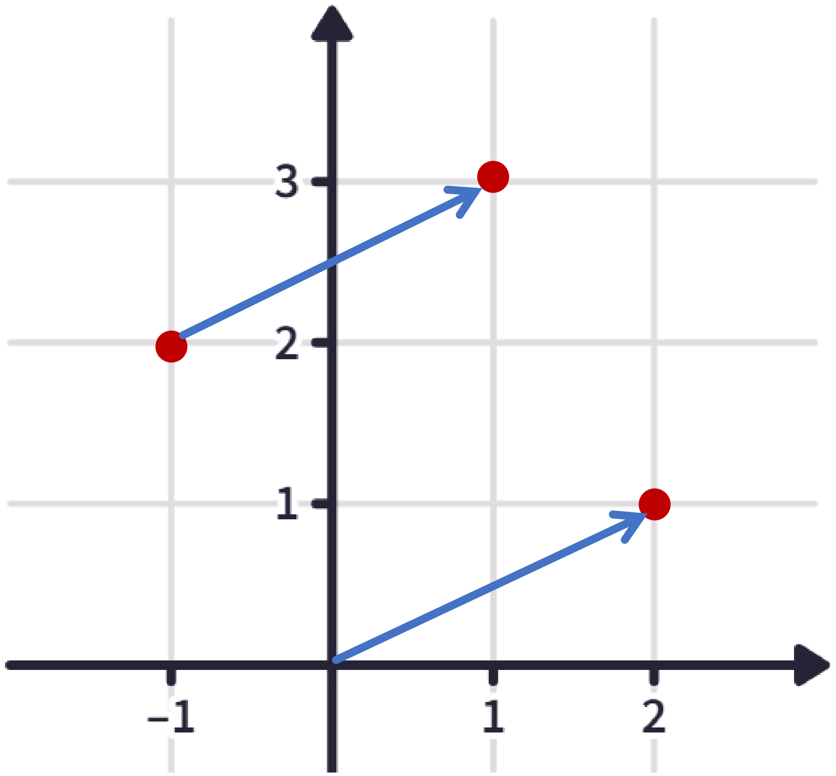


$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

VECTORS VIEWED GEOMETRICALLY

Example Draw the vector $\mathbf{v} = \langle 2,1 \rangle$ from the point $(-1,2)$.

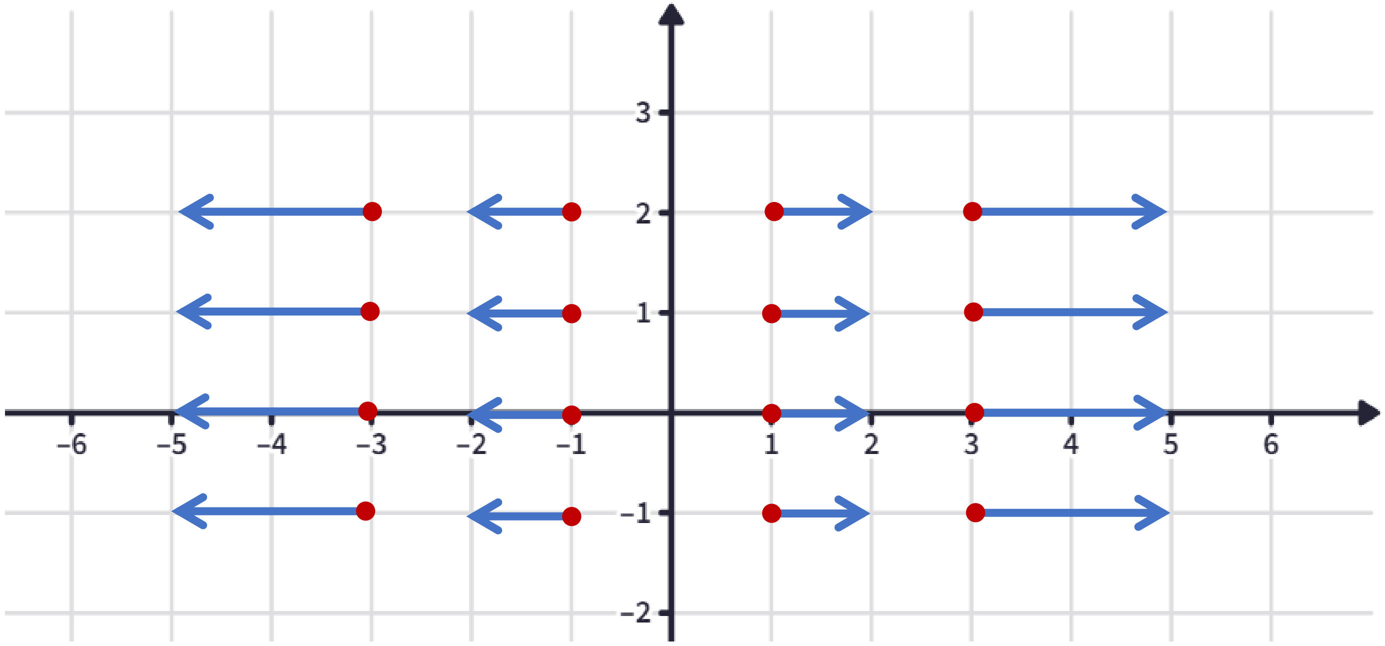
$$\text{End Point} = \text{Initial Point} + \mathbf{v}$$
$$= (-1,2) + (2,1)$$
$$= (1,3)$$



GRAPHICAL REPRESENTATIONS OF VECTOR FIELDS

Example Sketch some vectors in the vector field $\mathbf{F}(x, y) = x\,\mathbf{i}$.

$$\begin{aligned}\text{End Point} &= \text{Initial Point} + \mathbf{F}\\ &= (x, y) + (x, 0)\\ &= (2x, y)\end{aligned}$$



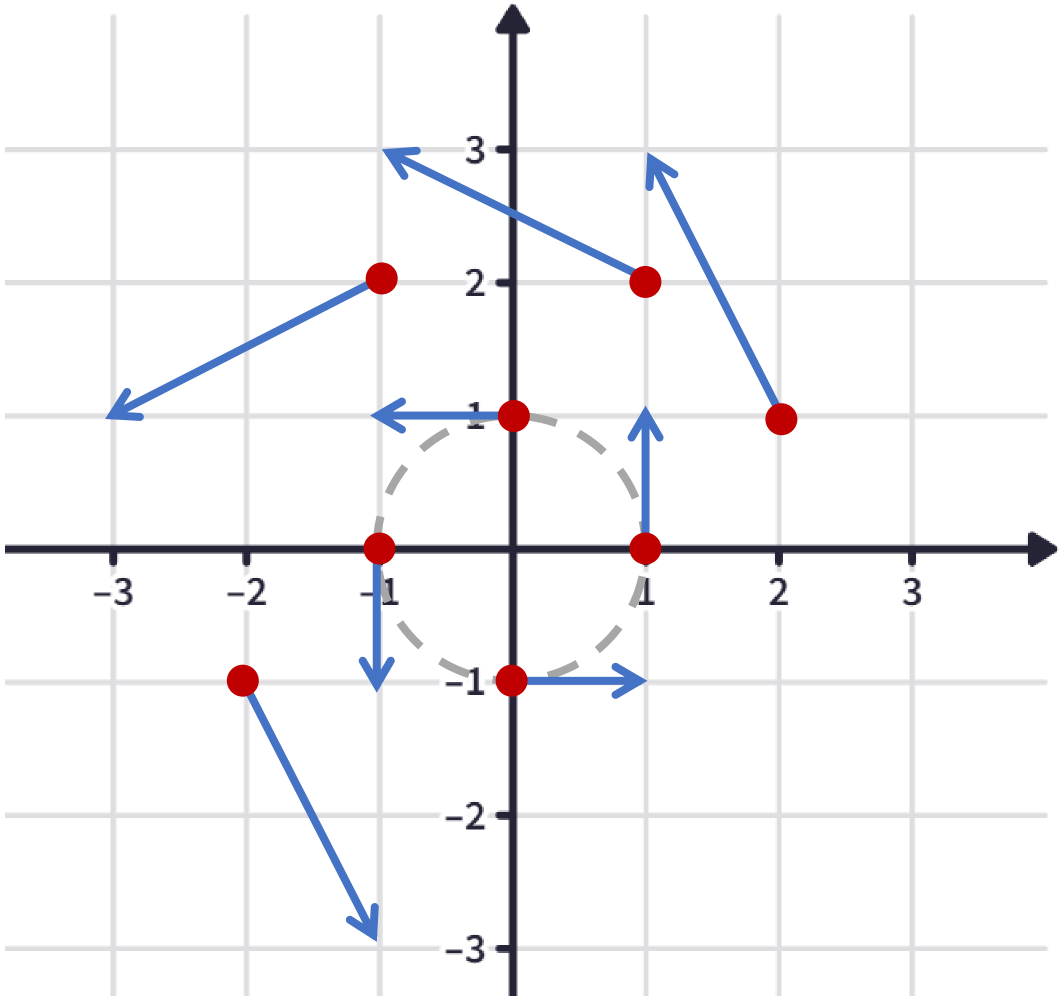
GRAPHICAL REPRESENTATIONS OF VECTOR FIELDS

Example Sketch some vectors in the vector field $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$.

$$\begin{aligned}\text{End Point} &= \text{Initial Point} + \mathbf{F}\\ &= (x, y) + (-y, x)\\ &= (x - y, x + y)\end{aligned}$$

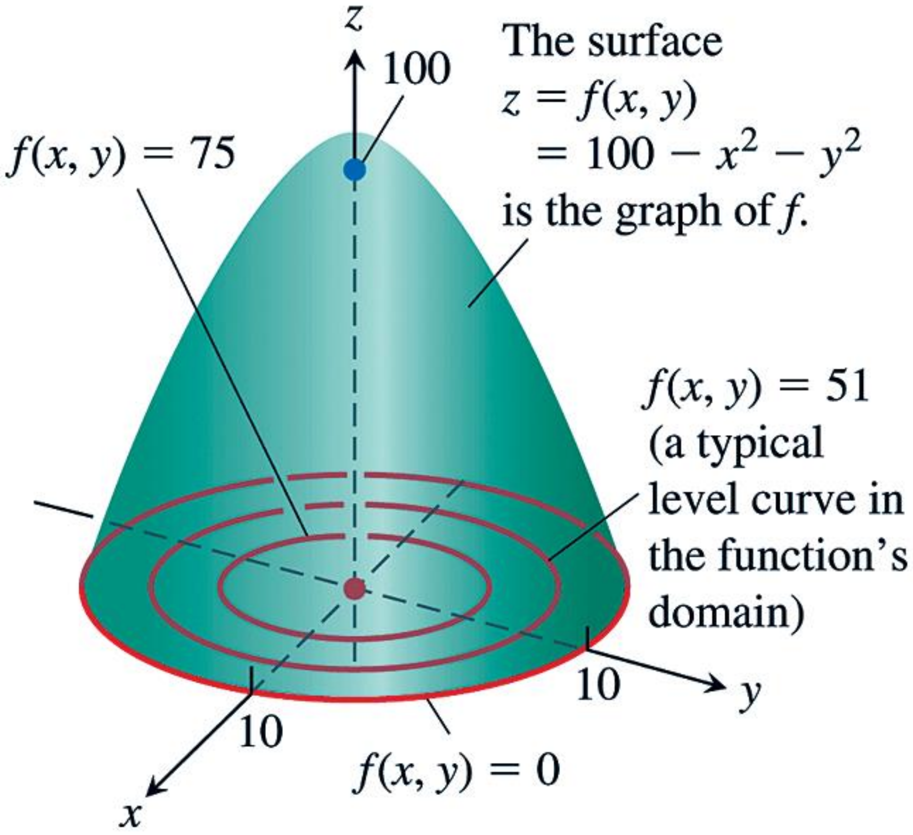
$$||\mathbf{F}|| = c \;\Rightarrow \sqrt{(-y)^2 + x^2} = c$$

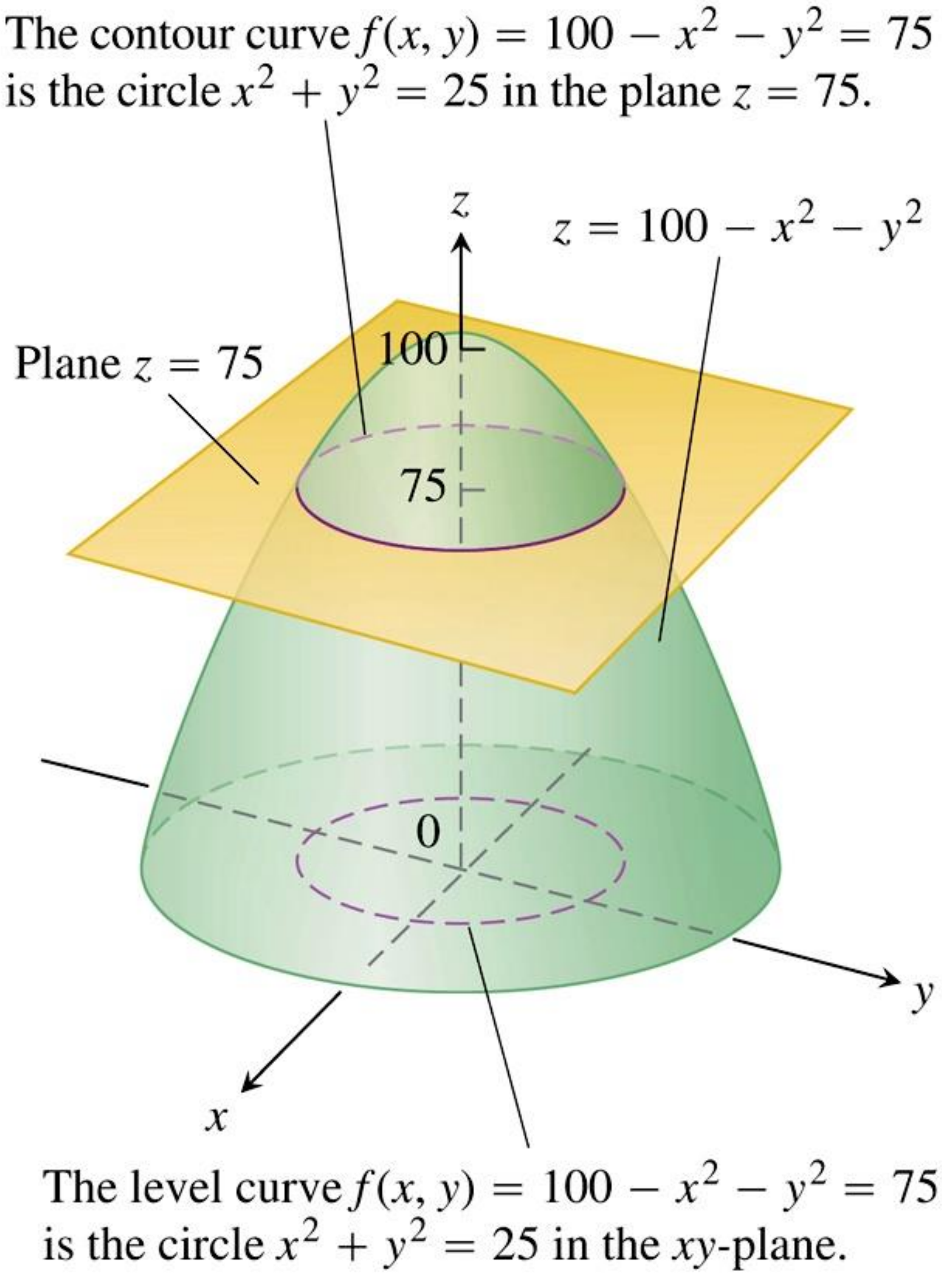
$$\Rightarrow x^2 + y^2 = c^2$$



LEVEL CURVES

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called a **level curve** of f.





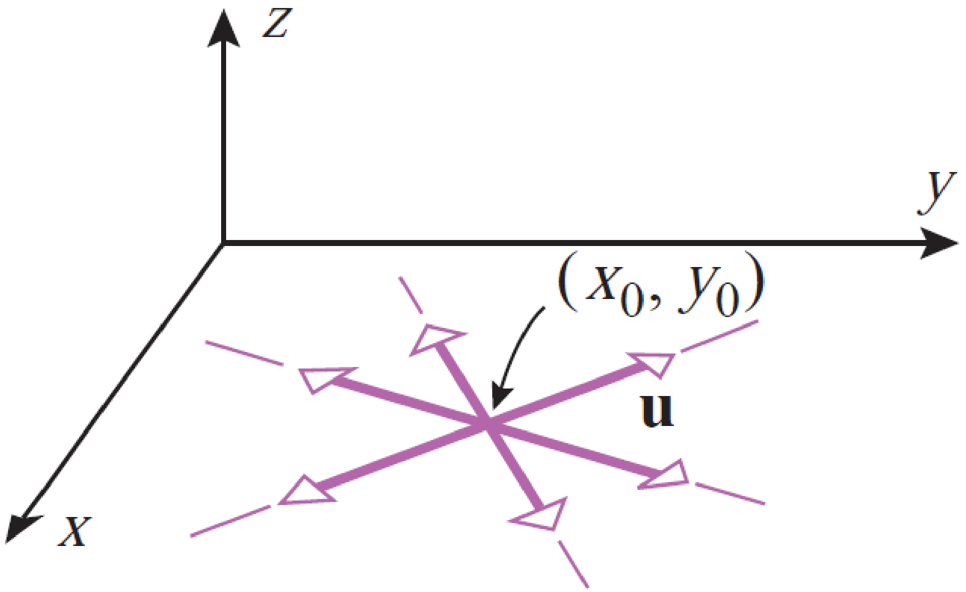
DIRECTIONAL DERIVATIVES

- To determine the slope at a point on a surface, you will define a new type of derivative called a *directional derivative*.

- To do this is to use a unit vector

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$$

that has its initial point at (x_0, y_0) and points in the desired direction.



$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

THE GRADIENT

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.

GRADIENT FIELDS

- If $f(x, y)$ is a function of two variables, then the gradient of f is given by

$$\nabla f = \langle f_x, f_y \rangle = f_x\,\mathbf{i} + f_y\,\mathbf{j}$$

- This formula defines a vector field in 2-space called **the gradient field of $\boldsymbol{f}$**.
- At each point in a gradient field where the gradient is nonzero, the vector points in the direction in which the rate of increase of f is maximum.

GRADIENT FIELDS

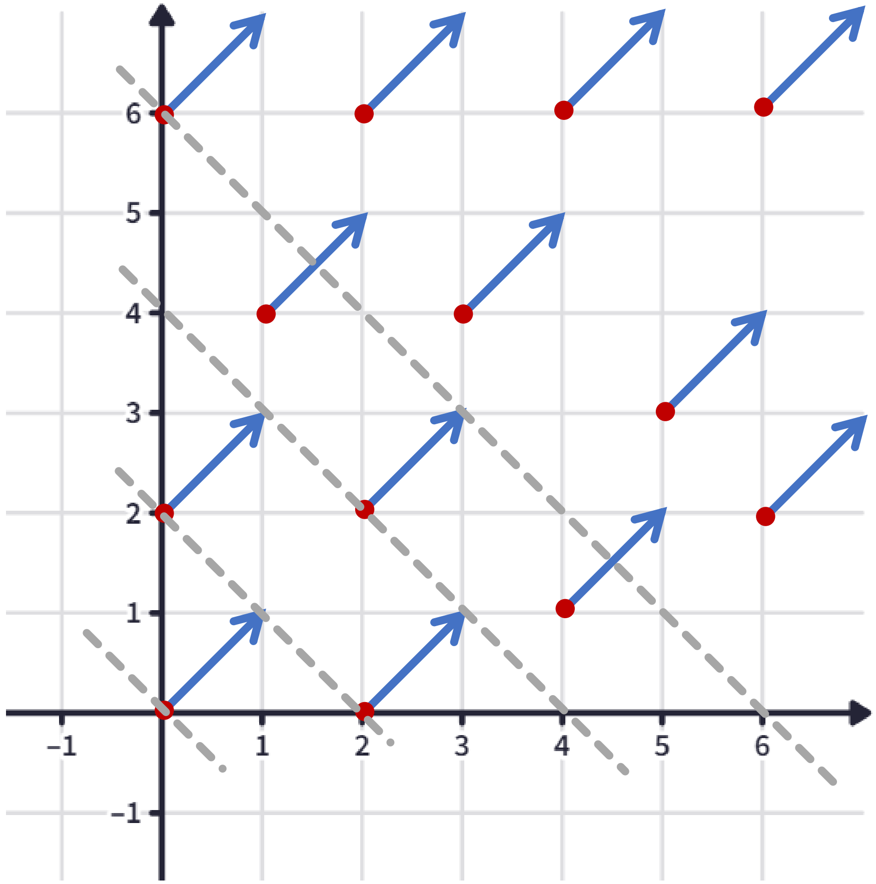
Example Sketch the gradient field of $f(x,y) = x + y$.

$$\nabla f = \mathbf{i} + \mathbf{j}$$

$$\begin{aligned} \text{End Point} &= \text{Initial Point} + \nabla f \\ &= (x,y) + (1,1) \\ &= (x+1, y+1) \end{aligned}$$

Draw some level curves.

Note that at each point, ∇f is normal to the level curve of f through the point.



CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

If $\mathbf{F}$ is an arbitrary vector field in $2-$space or $3-$space, we can ask whether it is the gradient field of some function f, and if so, how we can find f.

Definition: A vector field $\mathbf{F}$ in $2-$space or $3-$space is said to be **conservative** in a region if it is the gradient field for some function f in that region, that is, if

$$\mathbf{F} = \nabla f$$

The function f is called a **potential function for $\mathbf{F}$** in the region.

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

Example The vector field given by $\mathbf{F}(x, y) = 2x\,\mathbf{i} + y\,\mathbf{j}$ is conservative.

To see this, consider the potential function $f(x, y) = x^2 + \frac{1}{2}y^2$.

Because

$$\nabla f = 2x\,\mathbf{i} + y\,\mathbf{j} = \mathbf{F}$$

it follows that $\mathbf{F}$ is conservative.

TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Let $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open disk R. The vector field $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example Find α such that $\mathbf{F}(x, y) = (4x^2 + \alpha xy)\mathbf{i} + (3y^2 + 4x^2)\mathbf{j}$ is a gradient field (conservative).

$$M = 4x^2 + \alpha xy$$
$$N = 3y^2 + 4x^2$$

$\mathbf{F}$ is a gradient filed $\Leftrightarrow M_y = N_x$

$$\Leftrightarrow \alpha x = 8x$$
$$\Leftrightarrow \alpha = 8$$

TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Example Decide whether the vector field is conservative.

a) $\mathbf{F}(x, y) = \underbrace{x^2 y}_{M}\,\mathbf{i} + \underbrace{xy}_{N}\,\mathbf{j}$

$$\frac{\partial M}{\partial y} = x^2 \quad \neq \quad \frac{\partial N}{\partial x} = y$$

$\mathbf{F}$ is not conservative.

b) $\mathbf{F}(x, y) = \underbrace{2x}_{M}\,\mathbf{i} + \underbrace{y}_{N}\,\mathbf{j}$

$$\frac{\partial M}{\partial y} = 0 \quad = \quad \frac{\partial N}{\partial x} = 0$$

$\mathbf{F}$ is conservative.

FINDING A POTENTIAL FUNCTION FOR $\mathbf{F}(x, y)$

Example Find a potential function for $\mathbf{F}(x,y) = \underbrace{2xy}_{M}\,\mathbf{i} + \underbrace{(x^2 - y)}_{N}\,\mathbf{j}$.

There exists $f(x,y)$ such that $\mathbf{F} = \nabla f$.

$$2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j} = f_x\,\mathbf{i} + f_y\,\mathbf{j}$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x$$

$\therefore$ $\mathbf{F}$ is conservative.

$$f_x = 2xy \;\Rightarrow\; f = \int 2xy\,dx = x^2y + g(y)$$

$$f_y = x^2 - y \;\Rightarrow\; x^2 + g'(y) = x^2 - y$$

$$\Rightarrow\; g'(y) = -y$$

$$\Rightarrow\; g(y) = -\frac{1}{2}y^2 + c$$

$$\therefore\; f(x,y) = x^2y - \frac{1}{2}y^2 + c$$

THE ∇ OPERATOR

- The symbol ∇ that appears in the gradient expression ∇f has not been given a meaning of its own.
- It is often convenient to view ∇ as an operator.

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} \qquad \text{Plane}$$

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} \qquad \text{Space}$$

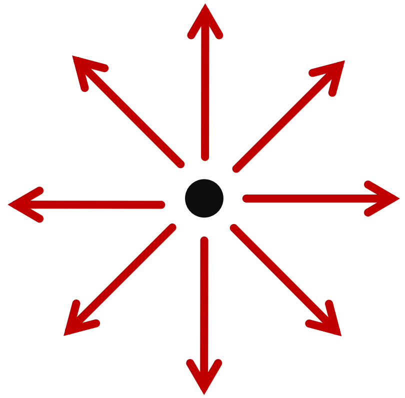
- We call it the **del operator**.

DIVERGENCE AND CURL

- We will now define two important operations on vector fields: the **divergence** and the **curl of the field**.
- These names originate in the study of **fluid flow**.
- Although we will focus only on their computation, seeing their physical meaning before is convenient.

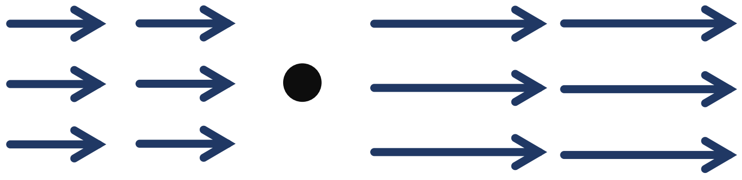
DIVERGENCE AND CURL

The **divergence** relates to the way in which fluid flows toward or away from a point.



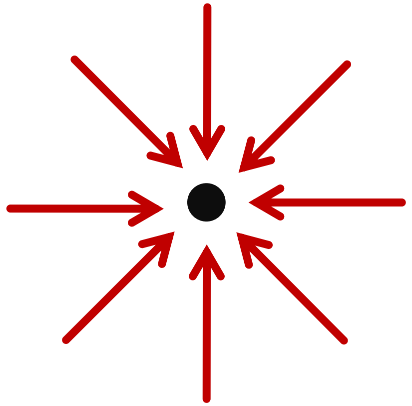
The field acting like a **source**.

$$\mathbf{div\,F}(x, y) > 0$$



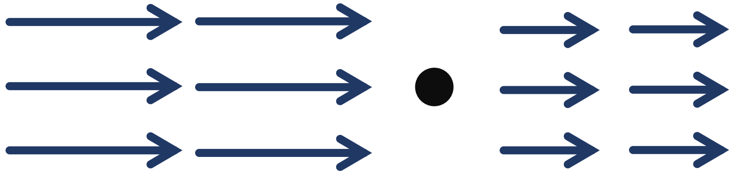
Slow flow in   Fast flow out

$$\mathbf{div\,F}(x, y) > 0$$



The field acting like a **sink**.

$$\mathbf{div\,F}(x, y) < 0$$

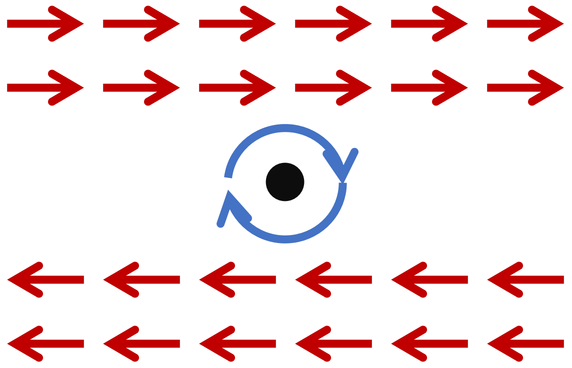


Fast flow in   Slow flow out

$$\mathbf{div\,F}(x, y) < 0$$

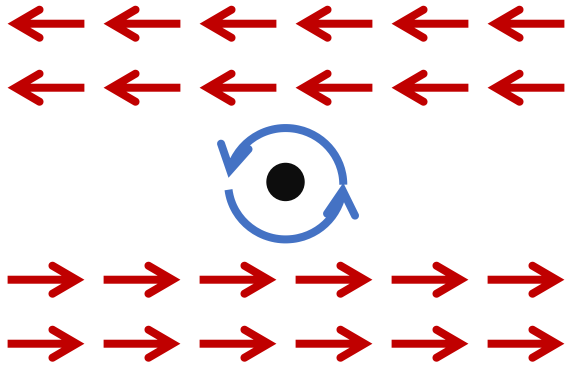
DIVERGENCE AND CURL

The **curl** relates to the rotational properties of the fluid at a point.



Clockwise Rotation

$\mathbf{Curl\ F}(x, y)$ is negative



Counter-Clockwise Rotation

$\mathbf{Curl\ F}(x, y)$ is positive

DEFINITION OF CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is

$$\text{curl}\,\mathbf{F}(x, y, z) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

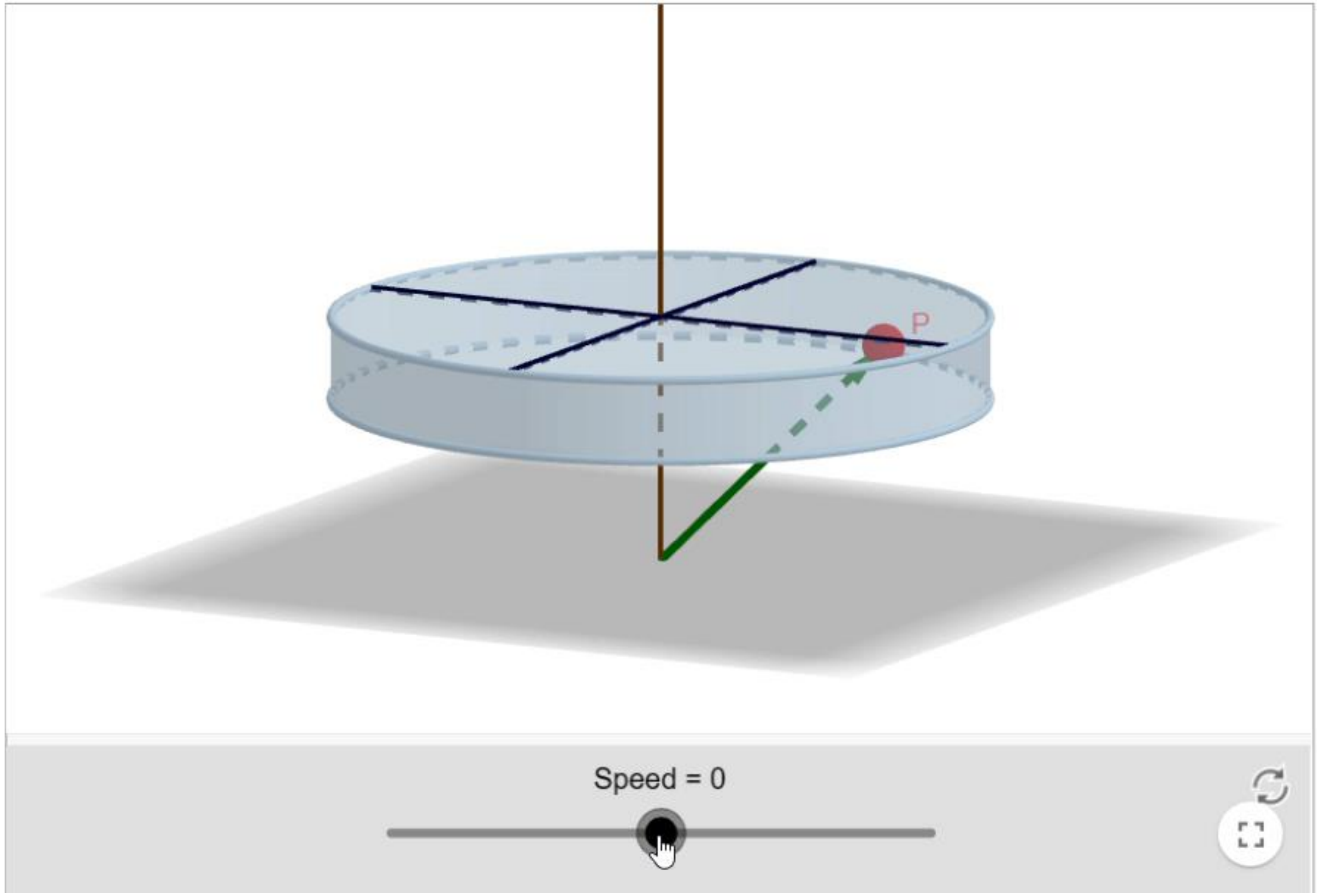
DEFINITION OF CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is

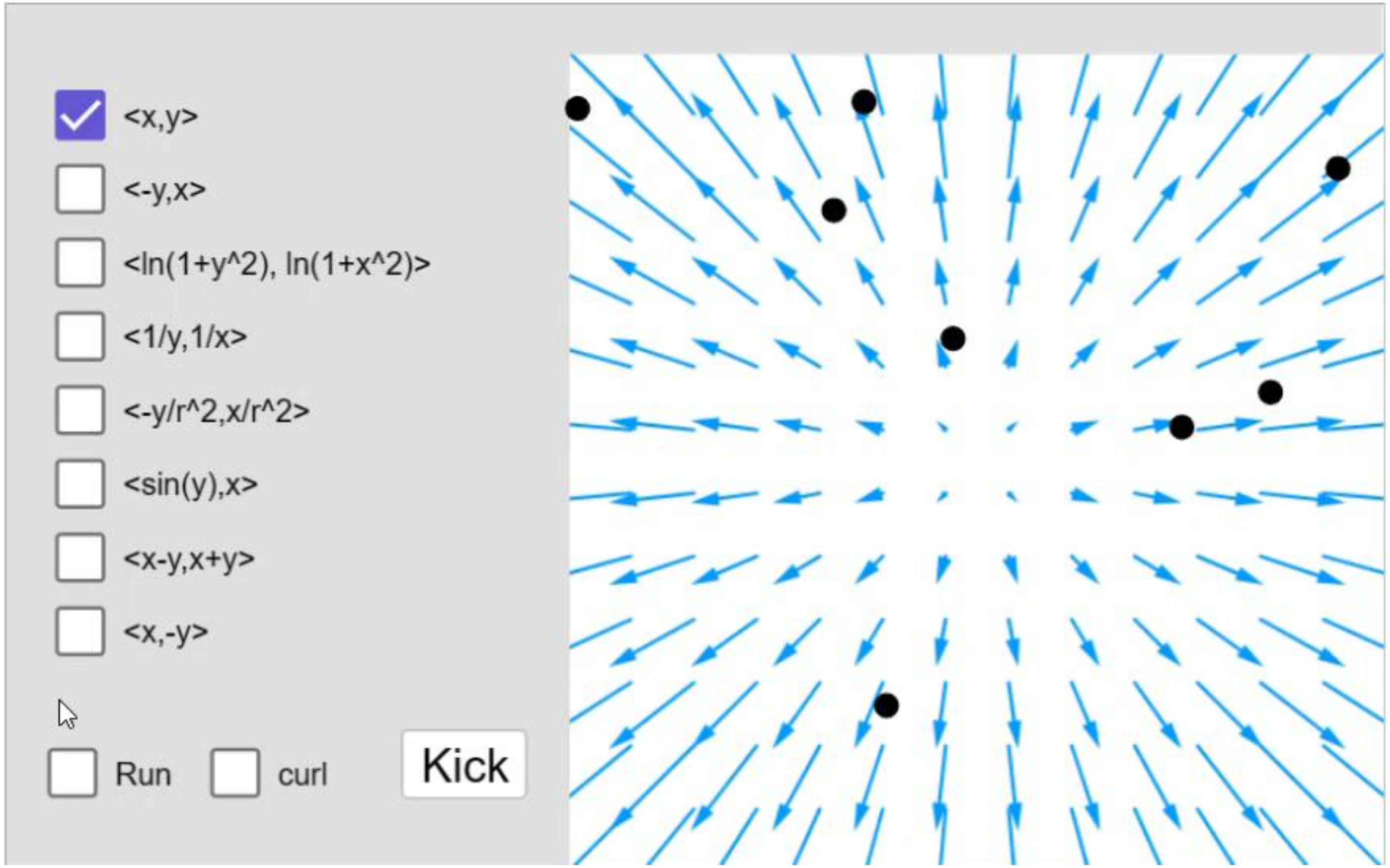
$$\text{curl}\ \mathbf{F}(x, y) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

DEFINITION OF CURL OF A VECTOR FIELD



DEFINITION OF CURL OF A VECTOR FIELD



DEFINITION OF CURL OF A VECTOR FIELD

Example Find curl $\mathbf{F}$ of the vector field

$$\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (2yz)\,\mathbf{k}$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & x^2 + z^2 & 2yz \end{vmatrix}$$

$$= (2z - 2z)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k}$$

$$= \mathbf{0}$$

TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Suppose that M, N, and P have continuous first partial derivatives in an open sphere Q in space. The vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if curl $\mathbf{F} = \mathbf{0}$.

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \quad , \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \quad , \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example The vector field $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (2yz)\,\mathbf{k}$ in the previous example is conservative because curl $\mathbf{F} = \mathbf{0}$.

TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Example Show that the vector field

$$\mathbf{F}(x, y, z) = \underbrace{x^3y^2z}_{M}\,\mathbf{i} + \underbrace{x^2z}_{N}\,\mathbf{j} + \underbrace{x^2y}_{P}\,\mathbf{k}$$

is not conservative.

$$\frac{\partial P}{\partial y} = x^2 \qquad \frac{\partial N}{\partial z} = x^2$$

$$\frac{\partial P}{\partial x} = 2xy \qquad \frac{\partial M}{\partial z} = x^3y^2$$

$\therefore$ $\mathbf{F}$ is not conservative

FINDING A POTENTIAL FUNCTION FOR $\mathbf{F}(x, y, z)$

Example Find a potential function

$$\mathbf{F}(x,y,z) = \underbrace{2xy}_{f_x}\,\mathbf{i} + \underbrace{(x^2+z^2)}_{f_y}\,\mathbf{j} + 2yz\,\mathbf{k}$$

From previous example, you know that the vector field is conservative. If $f(x,y,z)$ is a function such that $\mathbf{F} = \nabla f$, then

$$f_x = 2xy \;\Rightarrow\; f = \int 2xy\,dx = x^2y + g(y,z)$$

But $f_y = x^2 + z^2 \;\Rightarrow\; x^2 + g_y(y,z) = x^2 + z^2$

$$\Rightarrow g(y,z) = \int z^2 dy = z^2y + K(z)$$

$$\therefore f = x^2y + z^2y + K(z)$$

FINDING A POTENTIAL FUNCTION FOR $\mathbf{F}(x, y, z)$

Example Find a potential function

$$\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + \underbrace{2yz}_{f_z}\,\mathbf{k}$$

From previous example, you know that the vector field given by is conservative. If $f(x, y, z)$ is a function such that $\mathbf{F} = \nabla f$, then

$$\therefore\ f = x^2y + z^2y + K(z)$$

$$f_z = 2yz \ \Rightarrow\ 2yz + K'(z) = 2yz \ \Rightarrow\ K'(z) = 0 \ \Rightarrow\ K(z) = c$$

$$\therefore\ f(x, y, z) = x^2y + z^2y + c$$

DIVERGENCE OF A VECTOR FIELD

- The **divergence** of $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ in the plane is

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

- The **divergence** of $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ in the space is

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

DIVERGENCE OF A VECTOR FIELD

Example Find the divergence at the point $(2,1,-1)$ for the vector field

$$\mathbf{F}(x,y,z) = x^3y^2z\,\mathbf{i} + x^2z\,\mathbf{j} + x^2y\,\mathbf{k}$$

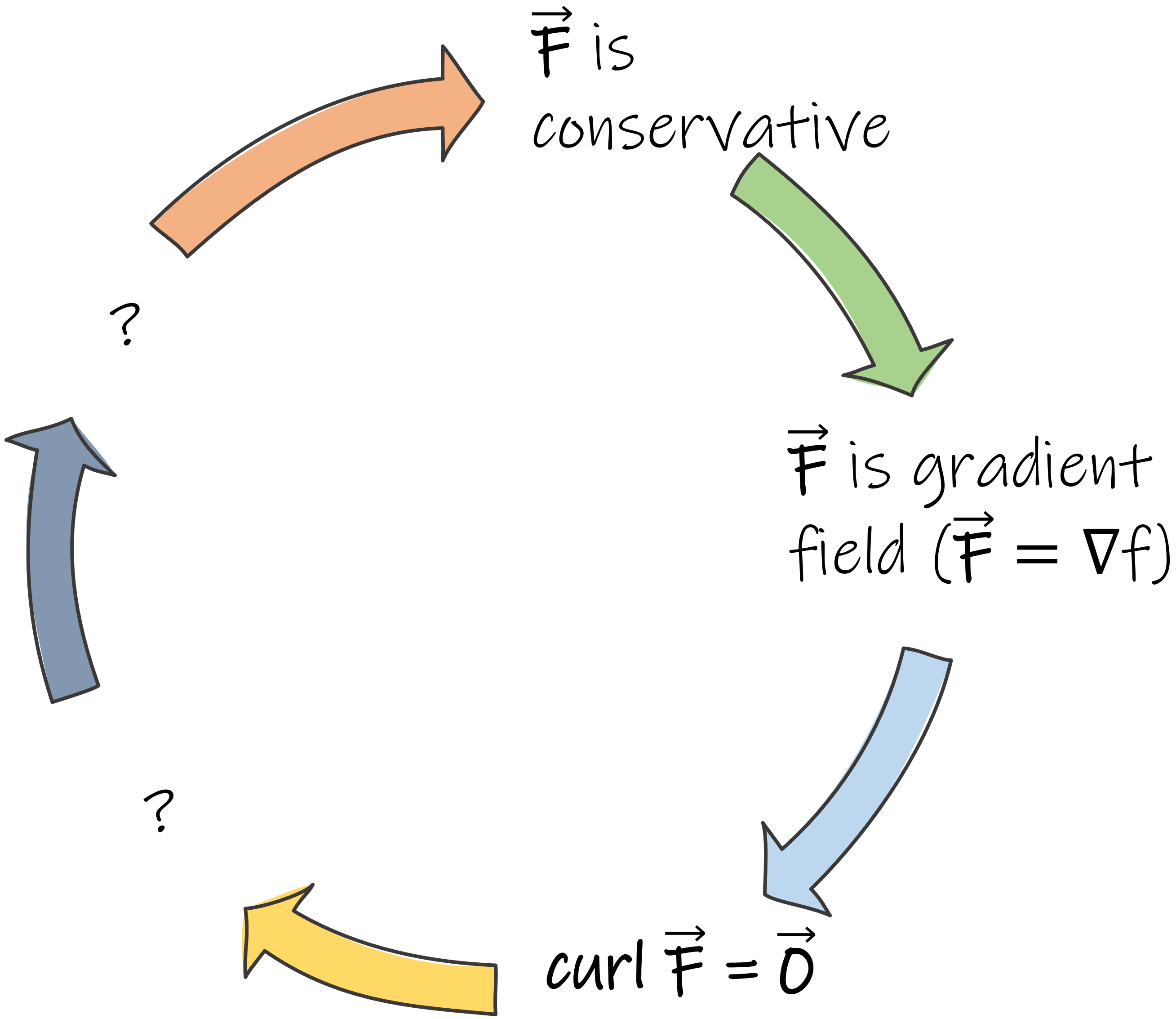
$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x^3y^2z) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(x^2y) = 3x^2y^2z$$

$$\text{div}\,\mathbf{F}(2,1,-1) = -12$$

Theorem If $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then

$$\text{div}(\text{curl}\,\mathbf{F}) = 0$$

EQUIVALENT STATEMENTS



Course: Calculus (4)

Chapter: [15]

TOPICS IN VECTOR CALCULUS

Section: [15.2]

LINE INTEGRALS

PARAMETRIC CURVES IN $3-$SPACE

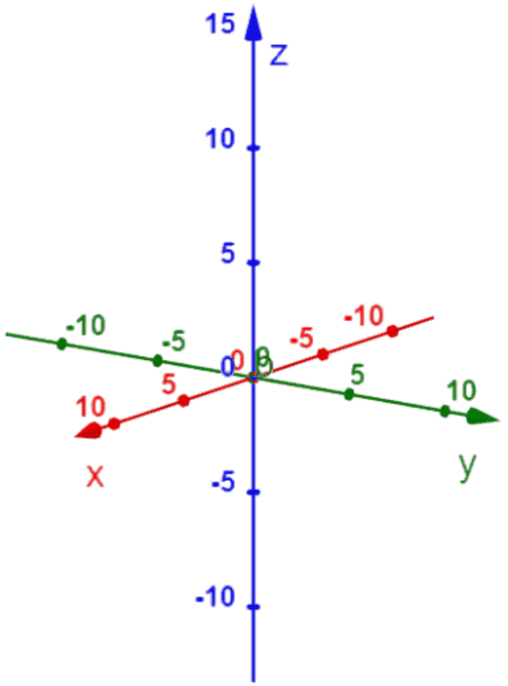
A *space curve C* is the set of all ordered triples (x, y, z) together with their defining parametric equations

$$x = f(t), y = g(t) \text{ and } z = h(t)$$

Example The Circular Helix

$$\begin{aligned} x &= 10\cos t \\ y &= 10\sin t \\ z &= t \end{aligned}$$

$$\begin{aligned} \mathbf{r}(t) &= 10\cos t\,\mathbf{i} + 10\sin t\,\mathbf{j} + t\mathbf{k} \\ &= \langle 10\cos t, 10\sin t, t\rangle \end{aligned}$$



SMOOTH PARAMETRIZATIONS

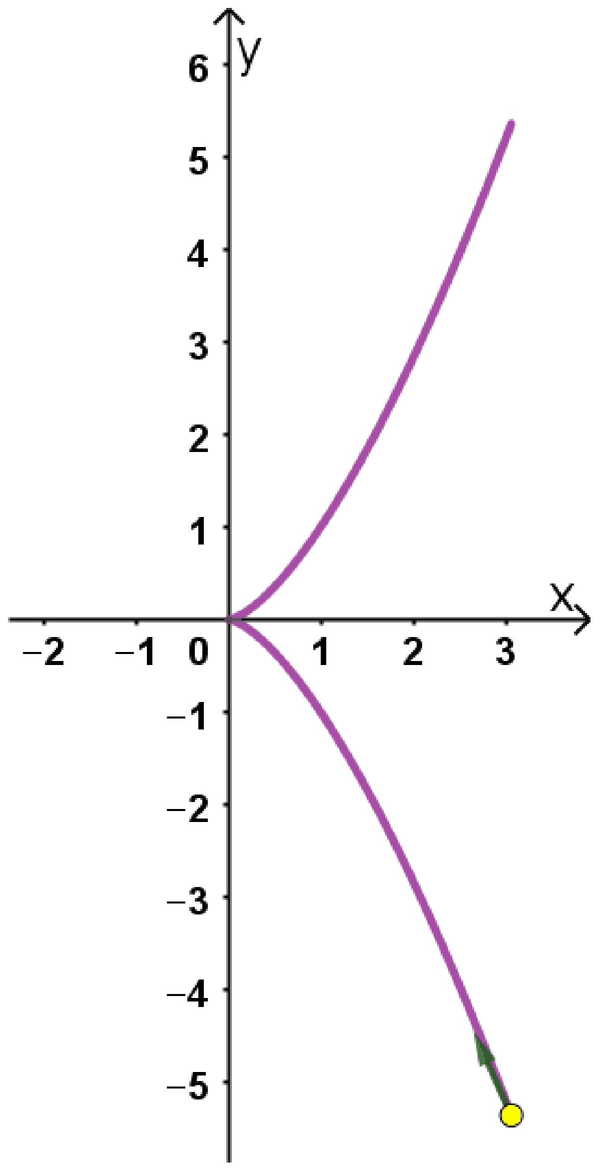
- We will say that a curve represented by $\mathbf{r}(t)$ is *smoothly parametrized* by $\mathbf{r}(t)$, or that $\mathbf{r}(t)$ is a smooth function of t if:
  - ✓ $\mathbf{r}'(t)$ is continuous, and
  - ✓ $\mathbf{r}'(t) \neq \mathbf{0}$ for any allowable value of t.
- Geometrically, this means that a smoothly parametrized curve can have no abrupt (مفاجئ) changes in direction as the parameter increases.

SMOOTH PARAMETRIZATIONS

Example Determine whether the vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is smooth.

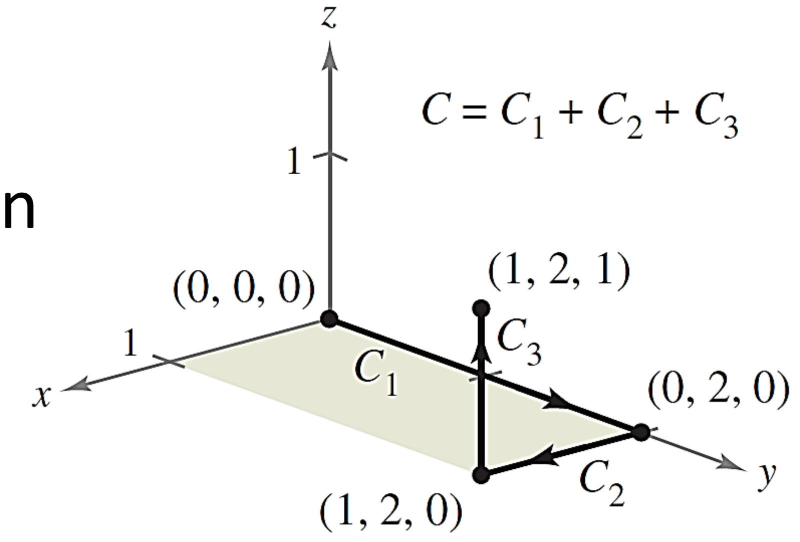
$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

- ✓ The components are continuous functions, and
- ✓ they are both equal to zero if $t = 0$.
- ✓ So, $\mathbf{r}(t)$ is NOT a smooth function at $t = 0$.



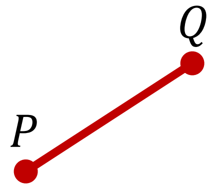
PIECEWISE SMOOTH PARAMETRIZATION

A curve C is **piecewise smooth** when the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which C is smooth.



Example Find a piecewise smooth parametrization of the graph of C shown in the figure.

C_1	C_2	C_3
$0 \leq t \leq 1$	$1 \leq t \leq 2$	$2 \leq t \leq 3$
$x(t) = 0$	$x(t) = t - 1$	$x(t) = 1$
$y(t) = 2t$	$y(t) = 2$	$y(t) = 2$
$z(t) = 0$	$z(t) = 0$	$z(t) = t - 2$

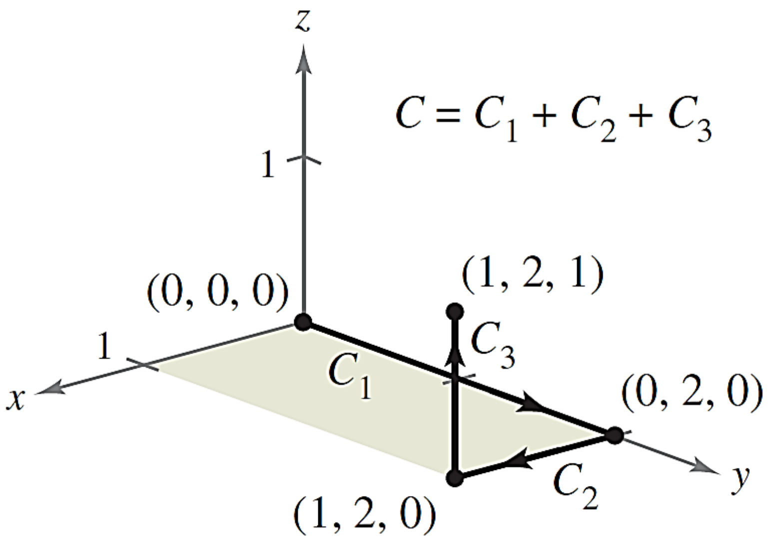


$$(1 - t)P + tQ$$
$$0 \leq t \leq 1$$

PIECEWISE SMOOTH PARAMETRIZATION

Example Find a piecewise smooth parametrization of the graph of C shown in the figure.

C_1	C_2	C_3
$0 \le t \le 1$	$1 \le t \le 2$	$2 \le t \le 3$
$x(t) = 0$	$x(t) = t - 1$	$x(t) = 1$
$y(t) = 2t$	$y(t) = 2$	$y(t) = 2$
$z(t) = 0$	$z(t) = 0$	$z(t) = t - 2$



$$C = \begin{cases} 2t\,\mathbf{j} & : \quad 0 \le t \le 1 \\ (t-1)\,\mathbf{i} + 2\,\mathbf{j} & : \quad 1 \le t \le 2 \\ \mathbf{i} + 2\,\mathbf{j} + (t-2)\,\mathbf{k} & : \quad 2 \le t \le 3 \end{cases}$$

ARC LENGTH FROM THE VECTOR VIEWPOINT

If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$, then its arc length ℓ from $t = a$ to $t = b$ is

$$\ell = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t = 0$ to $t = \pi$.

ARC LENGTH FROM THE VECTOR VIEWPOINT

$$\ell = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t = 0$ to $t = \pi$.

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\ell = \int_0^\pi \|\mathbf{r}'(t)\| dt$$

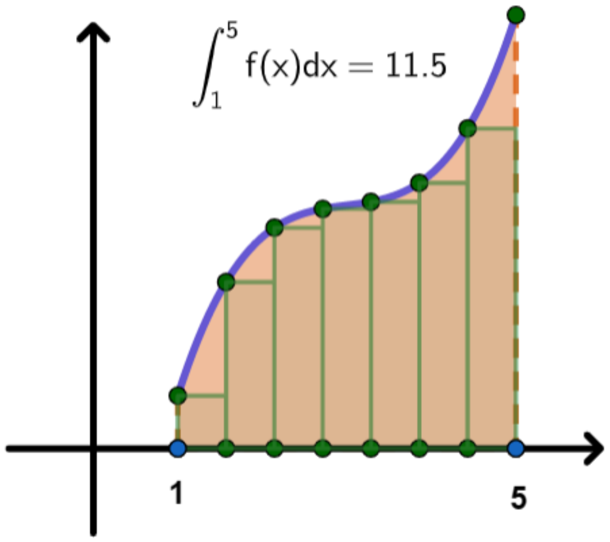
$$= \int_0^\pi \sqrt{2}\, dt$$

$$= \sqrt{2}\,\pi$$

LINE INTEGRALS

$$\int_a^b f(x)\,dx$$

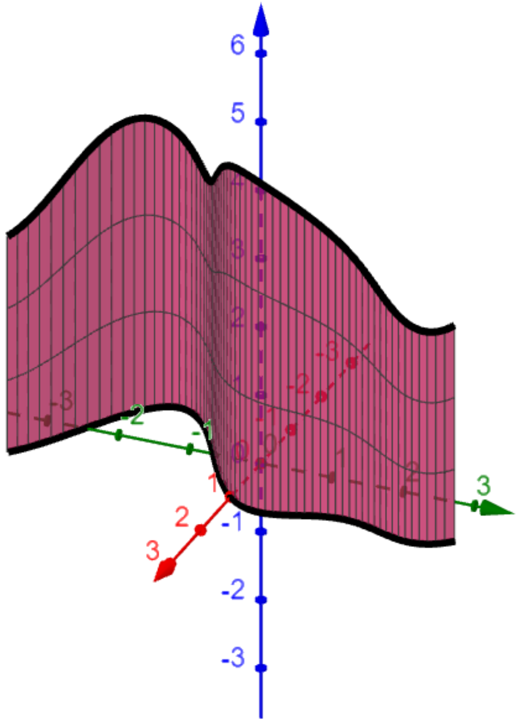
Integrate over interval $[a, b]$



$$\int_C f(x, y)\,ds$$

Integrate over a piecewise smooth curve C

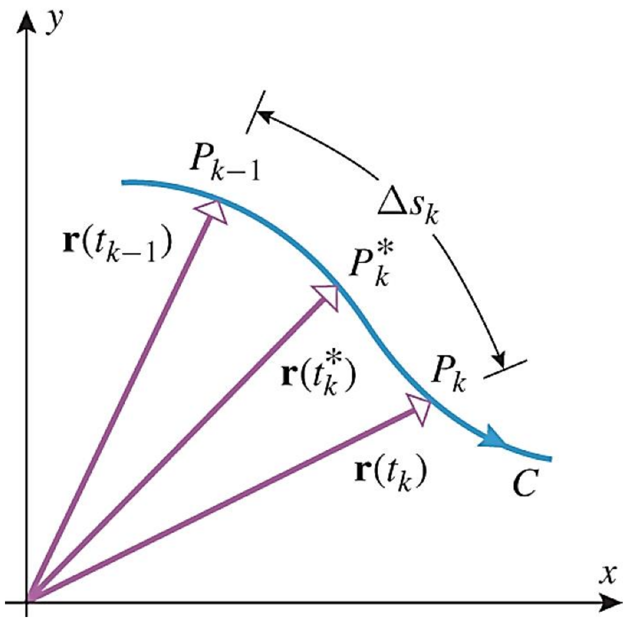
Line (Curve) Integral



EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ use the fact that

$$ds = \|\mathbf{r}'(t)\|dt = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$$



The arc length of C between points P_{k-1} and P_k is given by

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{r}'(t)\|\,dt = \|r'(t_k^*)\|\,\Delta t_k$$

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Therefore, if C is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \quad ; \quad t \in [a, b]$$

then

$$\int_C f(x, y)\, ds = \int_a^b f\big(x(t), y(t)\big)\, \|\mathbf{r}'(t)\| dt$$

$$= \int_a^b f\big(x(t), y(t)\big) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Similarly, if C is a curve in $3-$space that is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \quad ; \quad t \in [a, b]$$

then

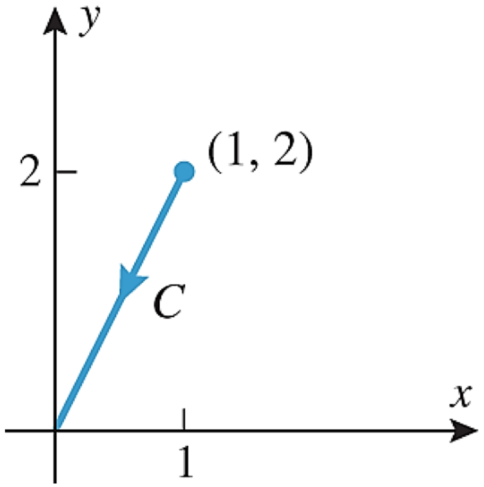
$$\int_C f(x, y, z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big)\, \|\mathbf{r}'(t)\| dt$$

$$= \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z(t)\big)^2} dt$$

Note: $\displaystyle\int_C 1\, ds = \int_a^b \|\mathbf{r}'(t)\|\, dt = \text{length of the curve } C$

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Example Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)ds$



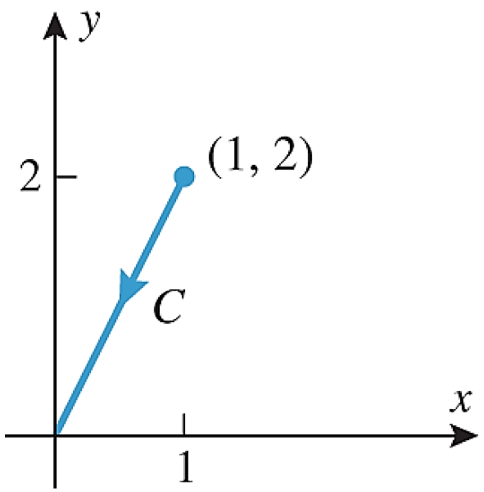
$$C = (1-t)(1,2) + t(0,0) \qquad t \in [0,1]$$
$$= (1-t, 2-2t) + (0,0)$$
$$= (1-t, 2-2t)$$
$$\mathbf{r}(t) = (1-t)\mathbf{i} + (2-2t)\mathbf{j}$$

$$x(t) = 1 - t$$
$$y(t) = 2 - 2t$$

$$\mathbf{r}'(t) = -\mathbf{i} - 2\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{5}$$

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Example Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)ds$



$$\int_C (1 + xy^2)\, ds = \int_0^1 (1 + (1 - t)(2 - 2t)^2)\|\mathrm{r}'(t)\| dt$$

$$= \int_0^1 (1 + 4(1 - t)^3)\sqrt{5} dt$$

$$= 2\sqrt{5}$$

$$x(t) = 1 - t$$

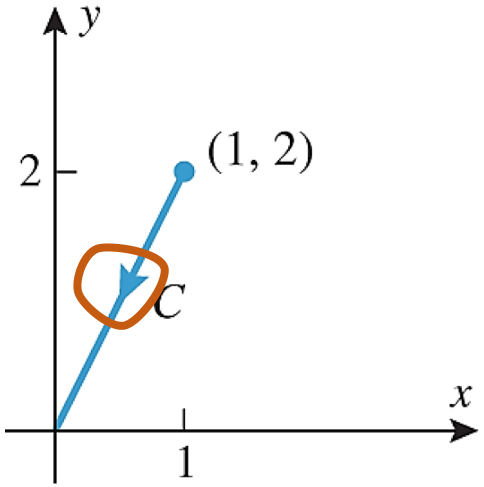
$$y(t) = 2 - 2t$$

$$\|\mathbf{r}'(t)\| = \sqrt{5}$$

$$t \in [0,1]$$

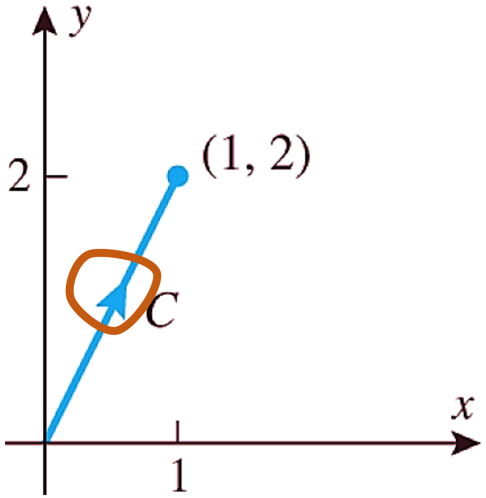
EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Example Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)ds$



$C = (1-t)(0,0) + t(1,2) = (t, 2t) \qquad t \in [0,1]$

$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} \qquad \mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} \qquad \|\mathbf{r}'(t)\| = \sqrt{5}$



$$\int_C (1 + xy^2)\, ds = \int_0^1 (1 + 4t^3)\|\mathrm{r}'(t)\|dt$$

$$= \int_0^1 (1 + 4t^3)\sqrt{5}dt = 2\sqrt{5}$$

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

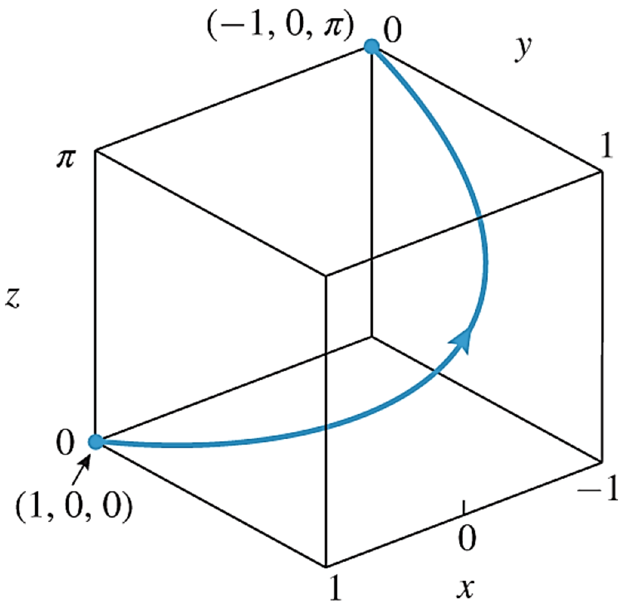
NOTE

- The integrals in the previous example agree, even though the corresponding parametrizations of C have opposite orientations.
- This illustrates the important result that the value of a line integral of f with respect to s along C does not depend on the parametrization of the line segment C ; **any smooth parametrization will produce the same value**.
- Later in this section we will see that **for line integrals of vector functions, the orientation of the curve is important**.

EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Example Evaluate the line integral $\int_C (xy + z^3)ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along the helix C that is represented by the parametric equations

$$x = \cos t \quad y = \sin t \quad z = t \quad t \in [0, \pi]$$



$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}$$

$$\int_C (xy + z^3)ds = \int_0^\pi (\cos t \sin t + t^3)\sqrt{2}dt = \frac{\sqrt{2}\,\pi^4}{4}$$

EVALUATING A LINE INTEGRAL OVER A PATH

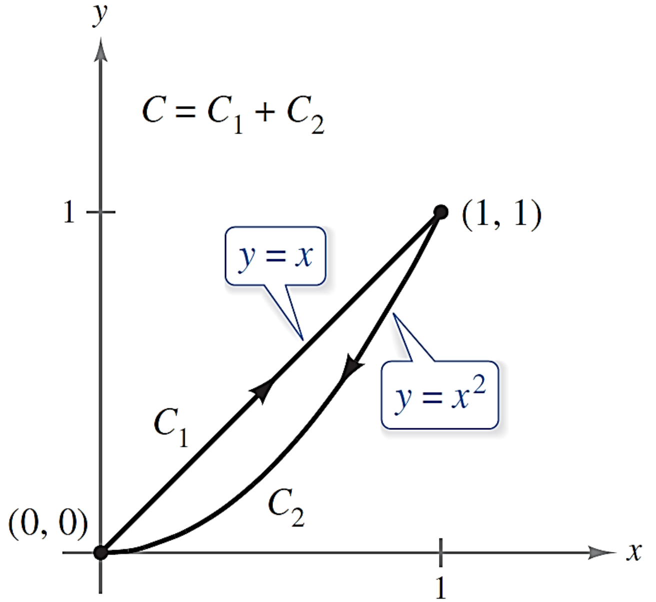
Let C be a path composed of smooth curves $C_1, C_2, \cdots, C_n$. If f is continuous on C, then it can be shown that

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds$$

Example Evaluate $\int_C xds$ where C is the piecewis smooth curve shown in the figure.

C_1: $x = t \quad y = t \quad t \in [0,1]$

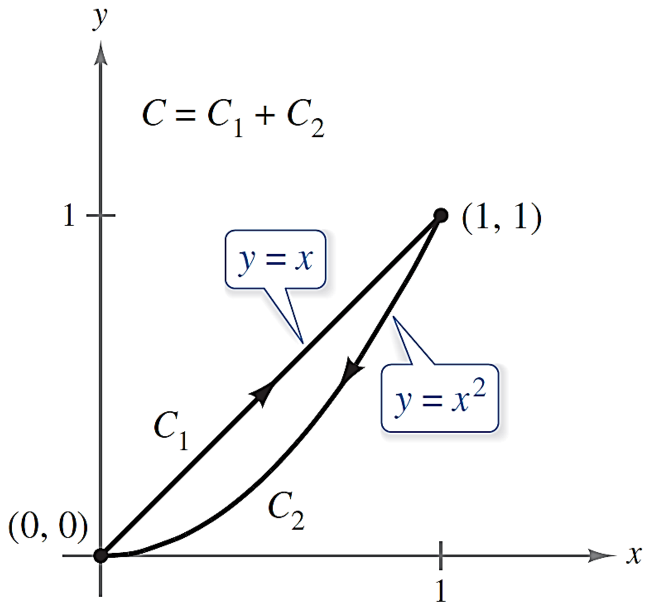
C_2: $x = 1 - t \quad y = (1-t)^2 \quad t \in [0,1]$



EVALUATING A LINE INTEGRAL OVER A PATH

Example Evaluate $\int_C xds$ where C is the piecewis smooth curve shown in the figure.

C_1: $x = t \quad y = t \quad t \in [0,1]$

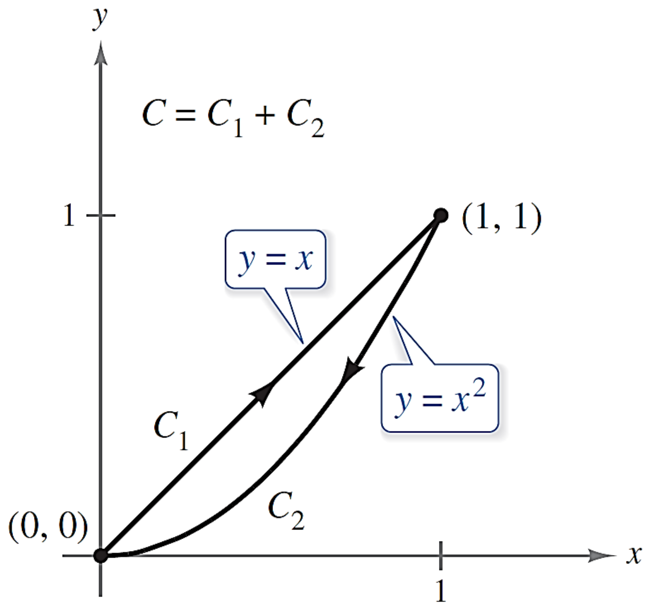


$$\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j}$$
$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{2}$$

$$\int_{C_1} xds = \int_0^1 t\sqrt{2}dt = \frac{1}{\sqrt{2}}$$

EVALUATING A LINE INTEGRAL OVER A PATH

Example Evaluate $\int_C xds$ where C is the piecewis smooth curve shown in the figure.



$$C_1: \quad x = t \quad y = t \quad t \in [0,1]$$
$$C_2: \quad x = 1 - t \quad y = (1-t)^2 \quad t \in [0,1]$$

$$\int_{C_2} xds = \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt$$

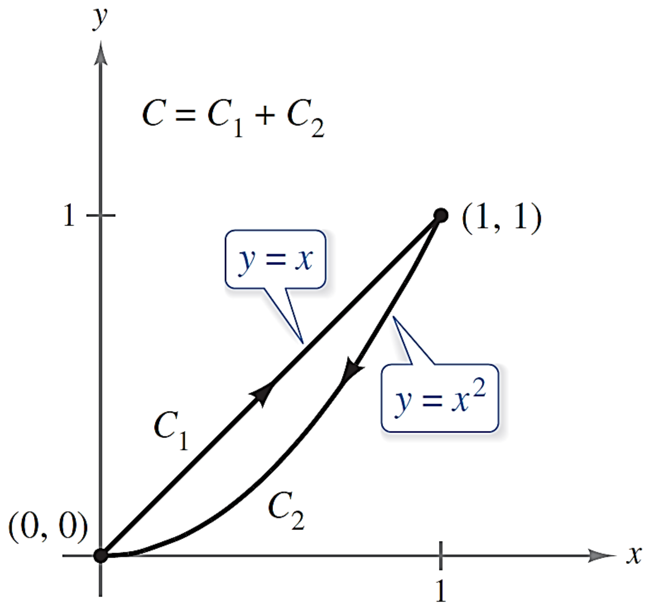
$$\mathbf{r}(t) = (1-t)\,\mathbf{i} + (1-t)^2\,\mathbf{j}$$
$$\mathbf{r}'(t) = -\mathbf{i} - 2(1-t)\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{1+4(1-t)^2}$$

$$= -\frac{1}{8}\int_5^1 \sqrt{u}\,du = \frac{1}{12}\left(5\sqrt{5}-1\right)$$

Let
$u = 1 + 4(1-t)^2$

EVALUATING A LINE INTEGRAL OVER A PATH

Example Evaluate $\int_C x ds$ where C is the piecewis smooth curve shown in the figure.



$$C_1: \quad x = t \quad y = t \quad t \in [0,1]$$

$$C_2: \quad x = 1 - t \quad y = (1-t)^2 \quad t \in [0,1]$$

$$\therefore \int_C x ds = \int_{C_1} x ds + \int_{C_2} x ds$$

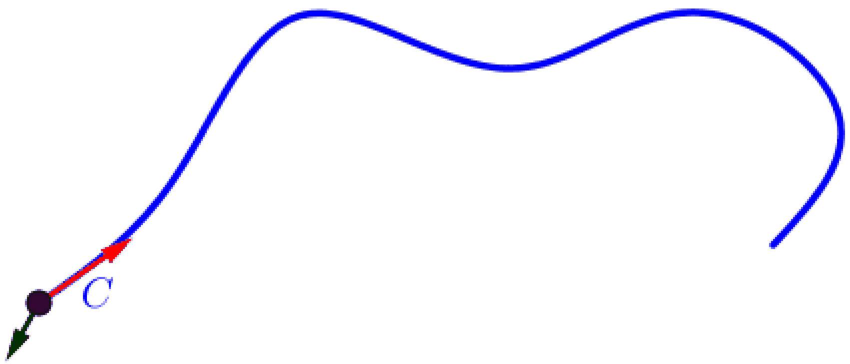
$$= \frac{1}{\sqrt{2}} + \frac{1}{12}\left(5\sqrt{5} - 1\right)$$

LINE INTEGRALS OF VECTOR FIELDS

- We can also consider integrating a vector field over a curve in the plane.
- One of the most important physical applications of line integrals is how to determine the **work** done by $\mathbf{F}$ in **moving a particle along a curve C**.
- The work W done by a *constant force* of magnitude F on a point that moves a distance d in a *straight* line in the direction of the force is $W = F\,d$.

LINE INTEGRALS OF VECTOR FIELDS

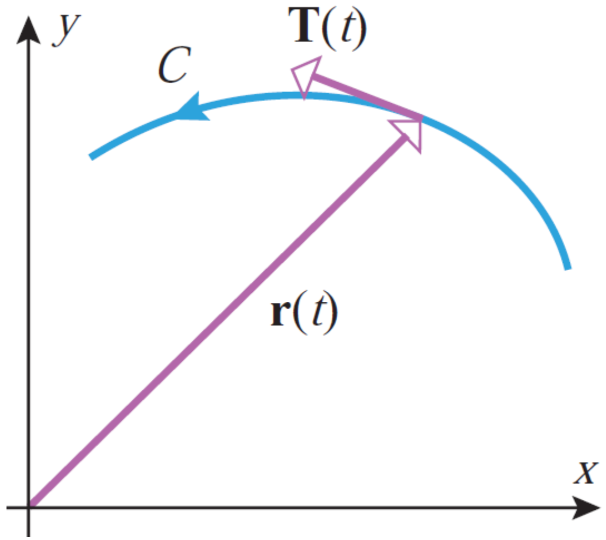
- Consider a force field $\mathbf{F}(x, y)$ and a piecewise continuous smooth curve $\mathbf{C}$.
- We wish to compute the work done by this force in moving a particle along a smooth curve $\mathbf{C}$.
- We divide $\mathbf{C}$ into sub-arcs with lengths Δs_i.
- To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is, $\mathbf{T}(t)$.



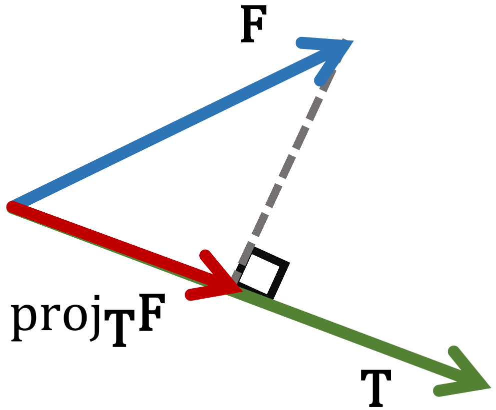
LINE INTEGRALS OF VECTOR FIELDS

We call $\mathbf{T}(t)$ the **unit tangent** vector to **C** at t, where

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

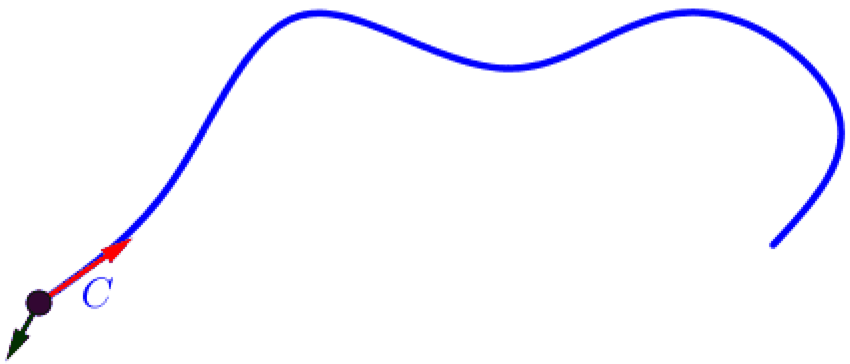


This means that at each point on C, you can consider the projection of the force vector $\mathbf{F}$ onto the unit tangent vector $\mathbf{T}$.



$$\mathrm{proj}_{\mathbf{T}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{T}}{\|\mathbf{T}\|^2}\mathbf{T}$$

$$= \underbrace{(\mathbf{F}\cdot\mathbf{T})}_{\textbf{FORCE}}\,\mathbf{T}$$



DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$ where $a \leq t \leq b$.

The line integral of $\mathbf{F}$ on C is given by

$$\mathbf{F} \cdot \mathbf{T}\, ds = \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt$$

$$= \mathbf{F} \cdot \mathbf{r}'(t) dt$$

$$= \mathbf{F} \cdot d\mathbf{r}$$

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big) \cdot \mathbf{r}'(t)\, dt$$

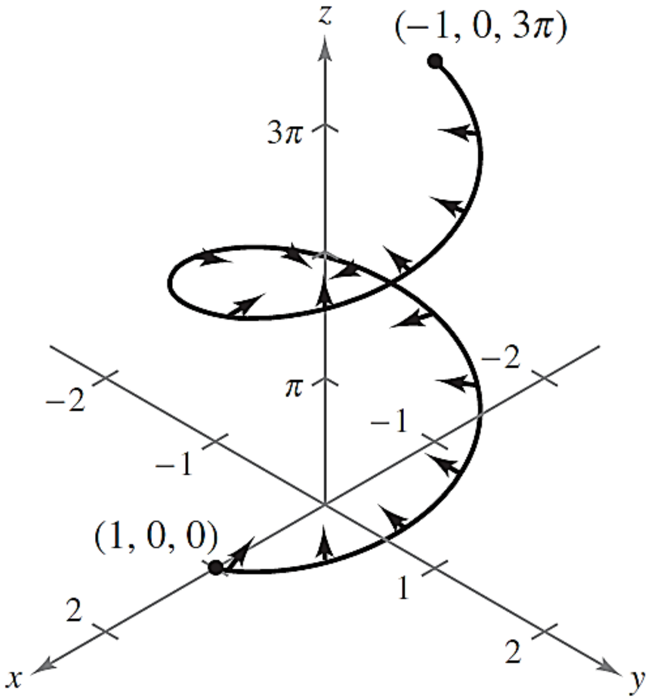
DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Example Find the work done by the force field

$$\mathbf{F}(x,y,z) = -\frac{1}{2}x\,\mathbf{i} - \frac{1}{2}y\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}$$

on a particle as it moves from the point $(1,0,0)$ to $(-1,0,3\pi)$ along the helix given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{\boxed{a}}^{\boxed{b}} \mathbf{F}\big(x(t),y(t),z(t)\big)\cdot \mathbf{r}'(t)\,dt$$



$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

$\cos t$	$\sin t$	t	
1	0	0	$t = 0$
-1	0	3π	$t = 3\pi$

DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Example Find the work done by the force field

$$\mathbf{F}(x, y, z) = -\frac{1}{2}x\,\mathbf{i} - \frac{1}{2}y\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}$$

on a particle as it moves from the point $(1,0,0)$ to $(-1,0,3\pi)$ along the helix given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

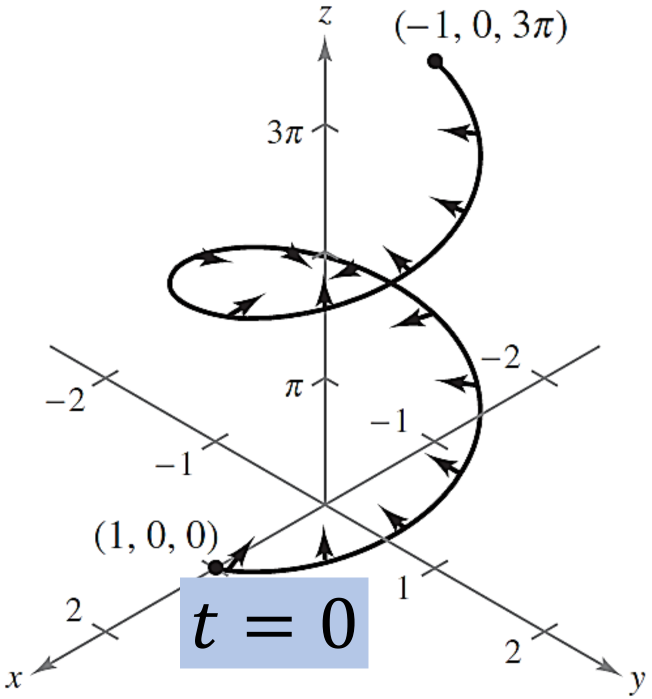
$t = 3\pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big)\cdot\mathbf{r}'(t)\,dt$$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

$$\mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \mathbf{k}$$

$$\mathbf{F}\big(x(t), y(t), z(t)\big) = -\frac{1}{2}\cos t\ \mathbf{i} - \frac{1}{2}\sin t\ \mathbf{j} + \frac{1}{4}\,\mathbf{k}$$



$t = 0$

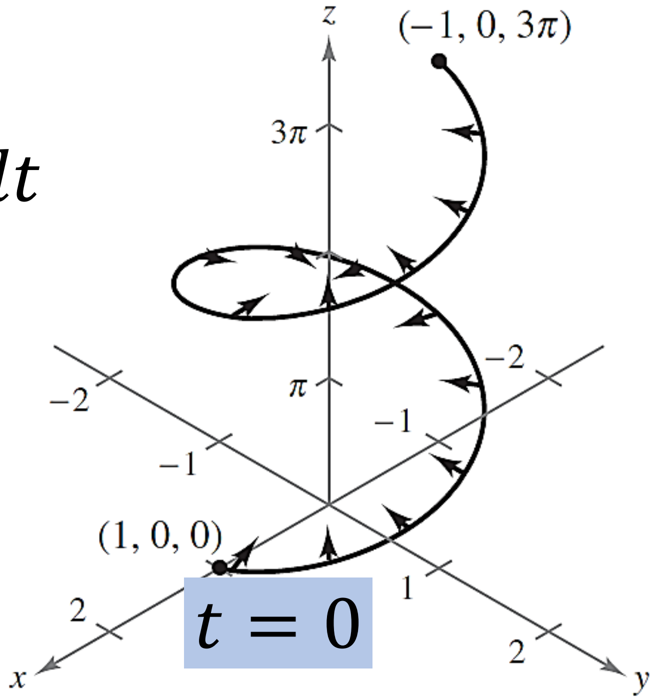
DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Example Find the work done by the force field

$$\mathbf{F}(x,y,z) = -\frac{1}{2}x\,\mathbf{i} - \frac{1}{2}y\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}$$

on a particle as it moves from the point $(1,0,0)$ to $(-1,0,3\pi)$ along the helix given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

$t = 3\pi$



$t = 0$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{3\pi} \left\langle -\frac{1}{2}\cos t\,, -\frac{1}{2}\sin t\,, \frac{1}{4}\right\rangle \cdot \langle -\sin t\,, \cos t\,, 1\rangle\, dt$$

$$= \int_0^{3\pi} \frac{1}{4}\, dt = \frac{3\pi}{4}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

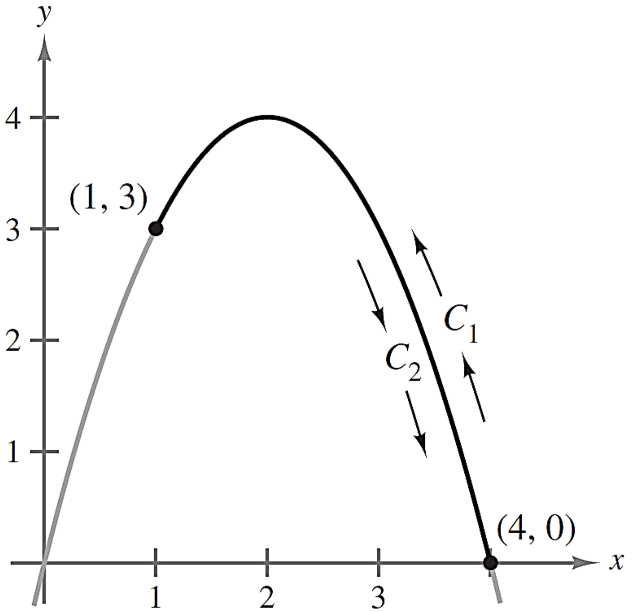
NOTE

- For line integrals of vector functions, the orientation of the curve C is important.
- If the orientation of the curve is reversed, the unit tangent vector $\mathbf{T}(t)$ is changed to $-\mathbf{T}(t)$, and you obtain

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Let $\mathbf{F}(x, y) = y\,\mathbf{i} + x^2\,\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each parabolic curve shown in the figure.



1. C_1: $\mathbf{r}_1(t) = (4 - t)\mathbf{i} + (4t - t^2)\mathbf{j} \quad 0 \le t \le 3$

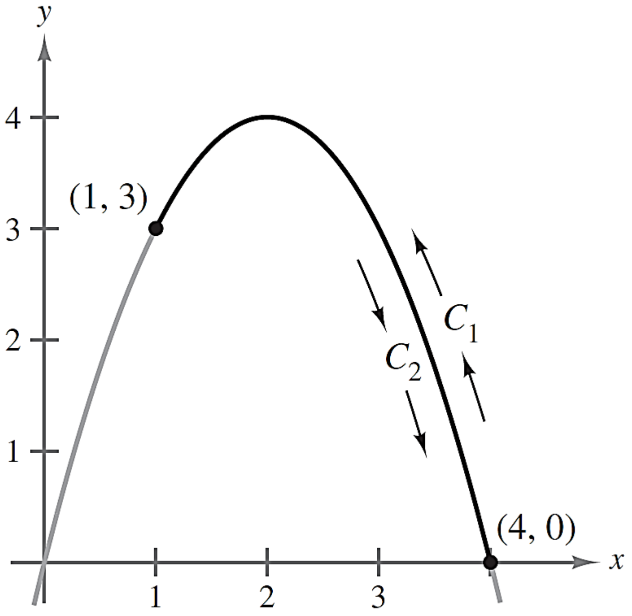
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^3 \mathbf{F}\big(x(t), y(t)\big) \cdot \mathbf{r}_1'(t)\,dt$$

$$= \int_0^3 \big((4t - t^2)\mathbf{i} + (4 - t)^2\,\mathbf{j}\big) \cdot (-\mathbf{i} + (4 - 2t)\mathbf{j})\,dt$$

$$= \int_0^3 (-2t^3 + 21t^2 - 68t + 64)\,dt = \frac{69}{2}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Let $\mathbf{F}(x, y) = y\,\mathbf{i} + x^2\,\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each parabolic curve shown in the figure.



2 C_2: $\mathbf{r}_2(t) = t\mathbf{i} + (4t - t^2)\mathbf{j} \quad 1 \le t \le 4$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r_2} = \int_1^4 \mathbf{F}\big(x(t), y(t)\big) \cdot \mathbf{r}_2'(t)\, dt$$

$$= \int_1^4 \big((4t - t^2)\mathbf{i} + t^2\,\mathbf{j}\big) \cdot (\mathbf{i} + (4 - 2t)\mathbf{j})\, dt$$

$$= \int_1^4 (-2t^3 + 3t^2 + 4t)\, dt \quad = -\frac{69}{2}$$

LINE INTEGRALS IN DIFFERENTIAL FORM

- A second commonly used form of line integrals is derived from the vector field notation used in Section 15.1.
- If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r}$ is often written as $M dx + N dy$.

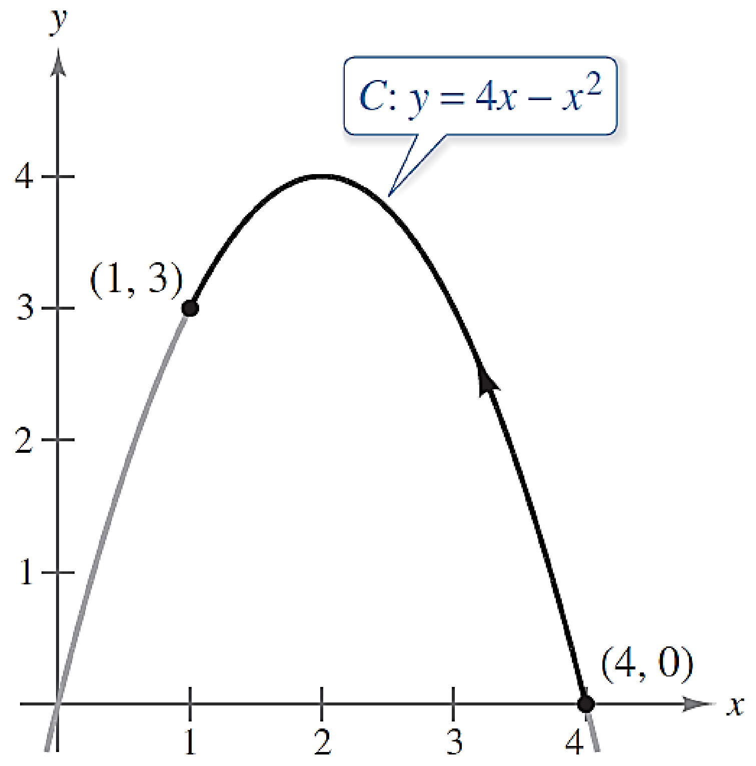
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_C (M\mathbf{i} + N\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j})\, dt$$

$$= \int_C \left(M \frac{dx}{dt} + N \frac{dy}{dt}\right) dt = \int_C (M dx + N dy)$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C (y\,dx + x^2 dy)$ where C is the parabolic arc given by $y = 4x - x^2$ from $(4,0)$ to $(1,3)$ as shown in the figure.

$$dy = (4 - 2x)dx$$



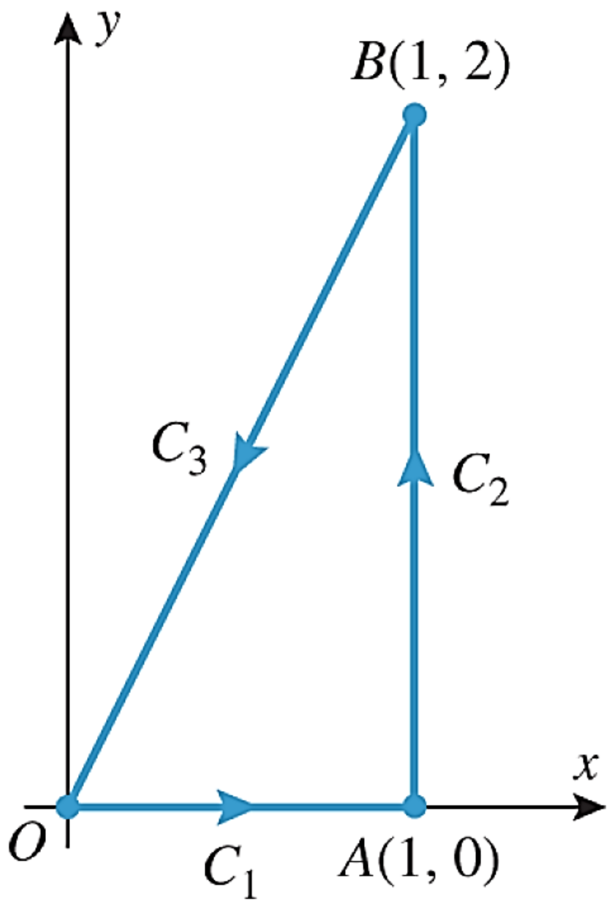
$$\int_C (y\,dx + x^2 dy) = \int_4^1 (4x - x^2)dx + x^2(4 - 2x)dx$$

$$= \int_4^1 (4x + 3x^2 - 2x^3)dx$$

$$= \frac{69}{2}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2y\,dx + x\,dy$ where C is the triangular path shown in the figure.



C_1: From $(0,0)$ to $(1,0)$ $\quad y = 0 \quad dy = 0$

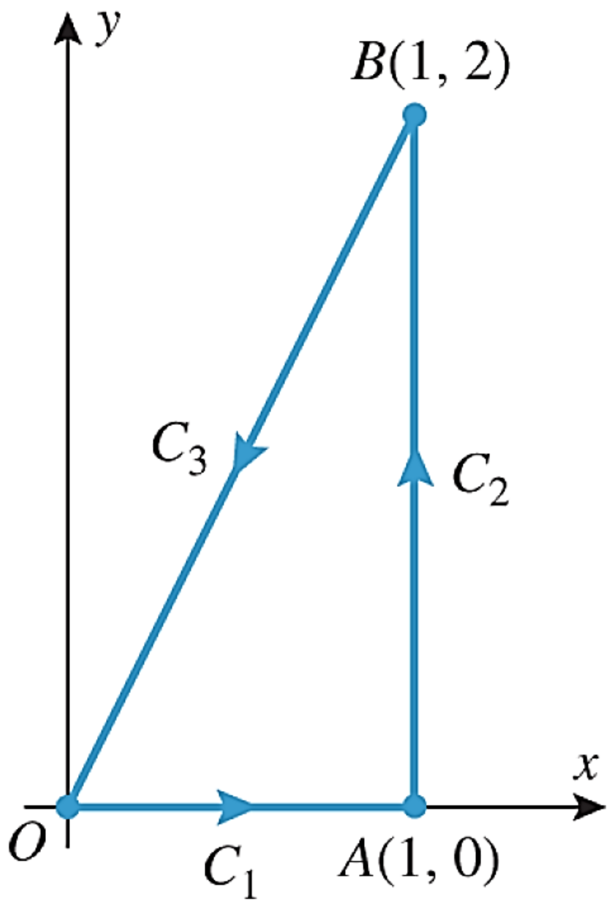
$$\int_{C_1} x^2y\,dx + x\,dy = \int_0^1 0\,dx = 0$$

C_2: From $(1,0)$ to $(1,2)$ $\quad x = 1 \quad dx = 0$

$$\int_{C_2} x^2y\,dx + x\,dy = \int_0^2 1\,dy = 2$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2ydx + xdy$ where C is the triangular path shown in the figure.



C_3: From $(1,2)$ to $(0,0)$ $y = 2x$ $dy = 2dx$

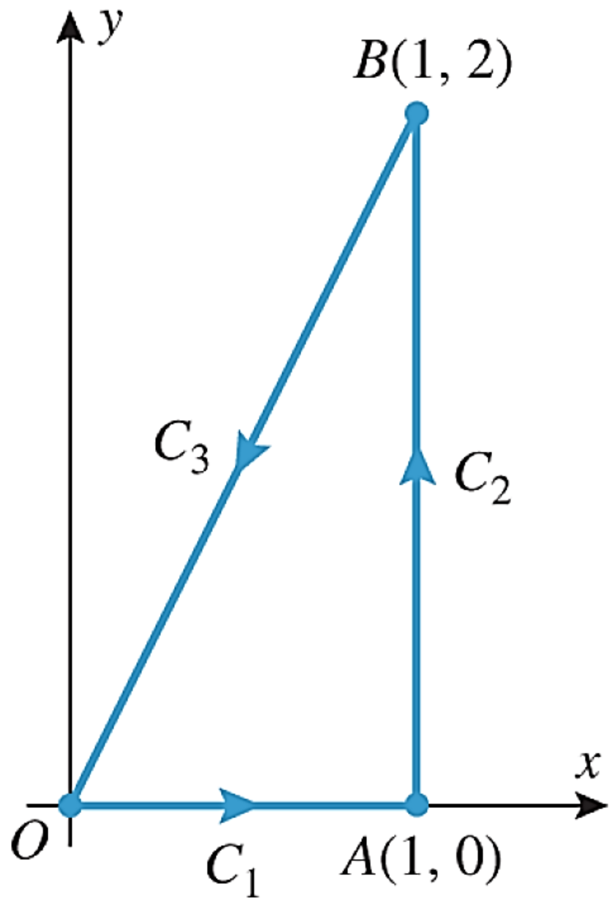
$$\int_{C_3} x^2ydx + xdy = \int_1^0 x^2(2x)dx + x(2dx)$$

$$= \int_1^0 (2x^3 + 2x)dx = -\frac{3}{2}$$

$$\therefore \int_C x^2ydx + xdy = 0 + 2 - \frac{3}{2} = \frac{1}{2}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2 y\,dx + x\,dy$ where C is the triangular path shown in the figure using parametric equations.



C_1: $x = t$ $dx = dt$ $y = 0$ $dy = 0$

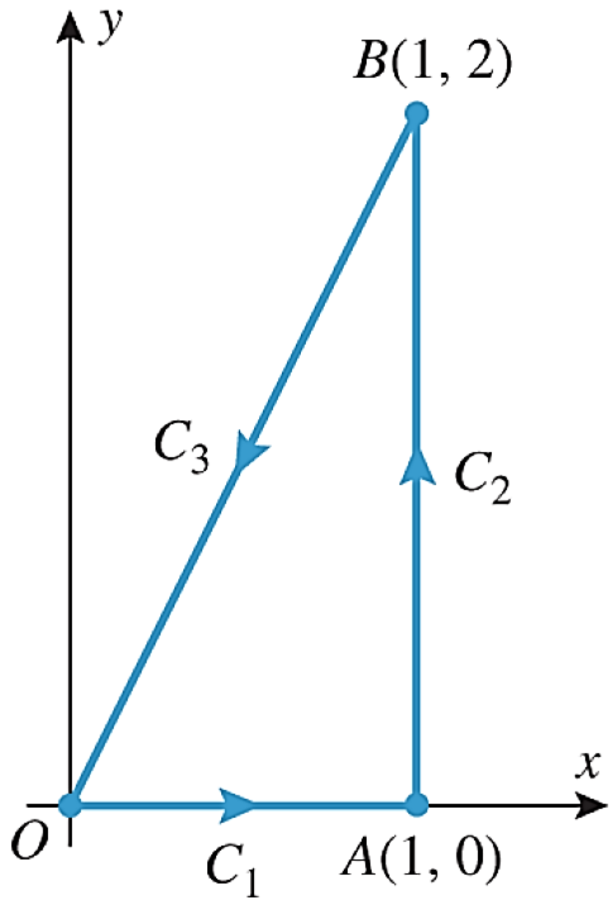
$$\int_{C_1} x^2 y\,dx + x\,dy = \int_0^1 (t^2)(0)(dt) + (t)(0) = 0$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2ydx + xdy$ where C is the triangular path shown in the figure using parametric equations.

C_2: $x = 1$ $dx = 0$ $y = 2t$ $dy = 2dt$

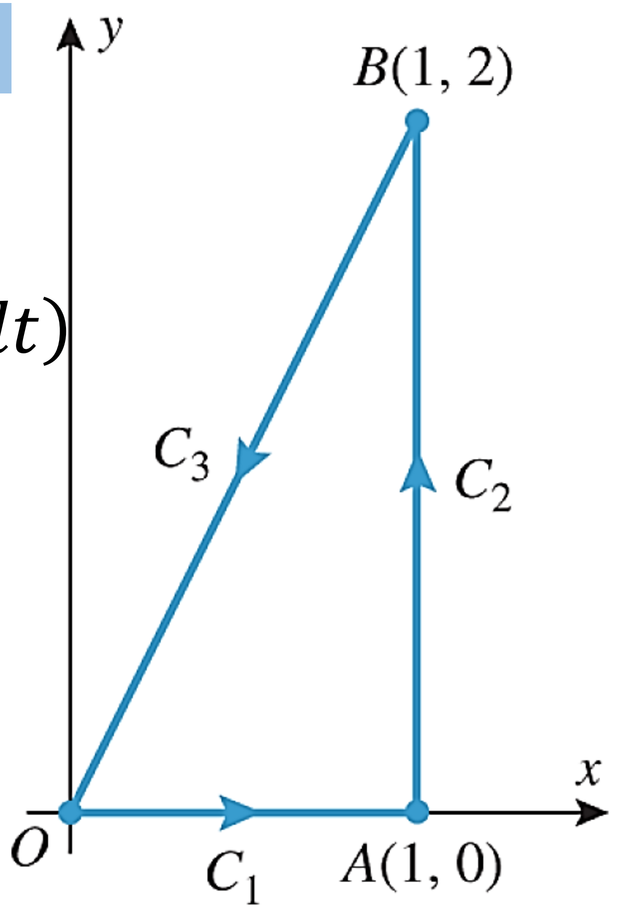
$$\int_{C_2} x^2ydx + xdy = \int_0^1 (1^2)(2t)(0) + (1)(2dt) = 2$$



ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2 y dx + x dy$ where C is the triangular path shown in the figure using parametric equations.

C_3: $x = 1 - t$ $dx = -dt$ $y = 2 - 2t$ $dy = -2dt$



$$\int_{C_3} x^2 y dx + x dy = \int_0^1 (1-t)^2(2-2t)(-dt) + (1-t)(-2dt)$$

$$= \int_0^1 -2(1-t)^3 dt - \int_0^1 2(1-t) dt$$

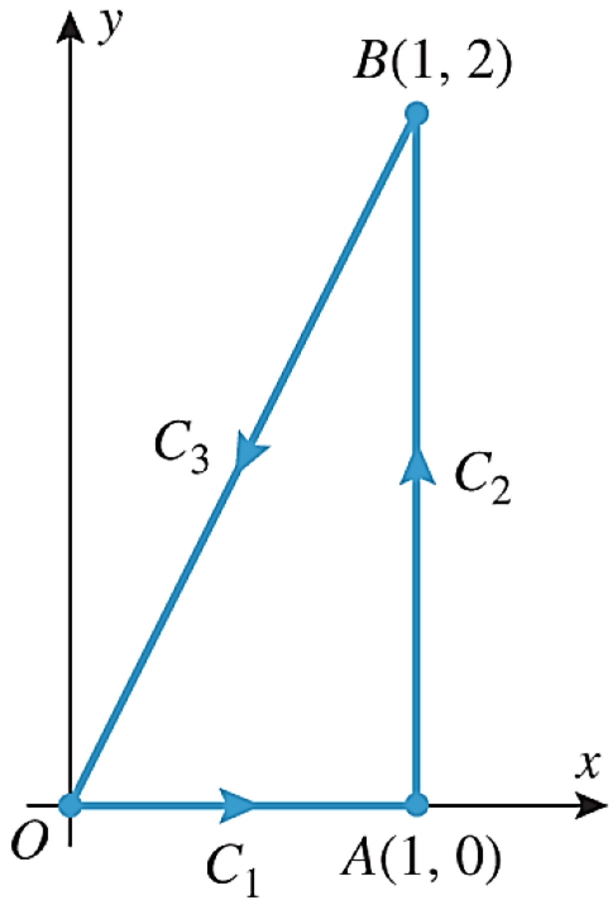
$$= -\frac{1}{2} - 1 = -\frac{3}{2}$$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example Evaluate $\int_C x^2ydx + xdy$ where C is the triangular path shown in the figure using parametric equations.

$$\int_C x^2ydx + xdy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= 0 + 2 - \frac{3}{2} = \frac{1}{2}$$



Course: Calculus (4)

Chapter: [15]

TOPICS IN VECTOR CALCULUS

Section: [15.3]

INDEPENDENCE OF PATH; CONSERVATIVE VECTOR FIELDS

LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x,y)=\frac{1}{2}xy\mathbf{i}+\frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.

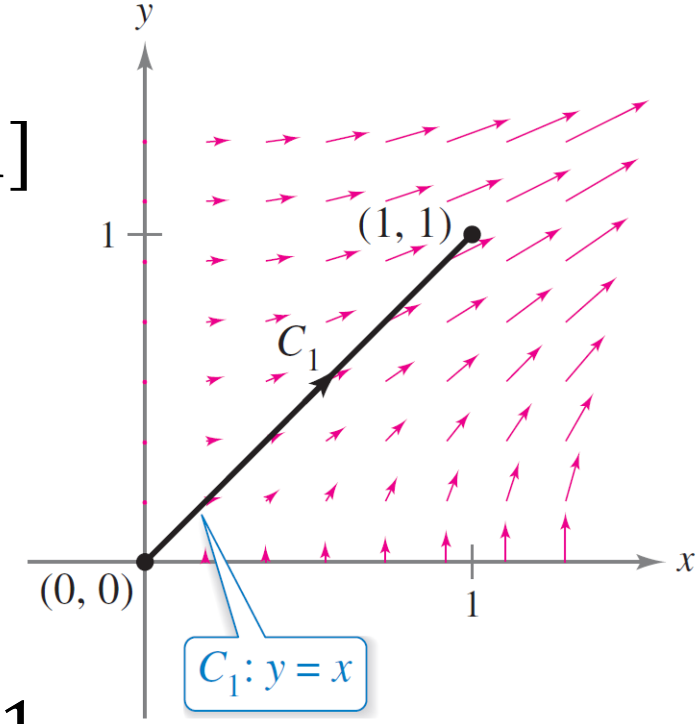
1

$$C_1 = (1-t)(0,0)+t(1,1) = (t,t) \quad t\in[0,1]$$

$$\mathbf{r}_1(t)=t\mathbf{i}+t\mathbf{j} \qquad \mathbf{r}_1'(t)=\mathbf{i}+\mathbf{j}$$

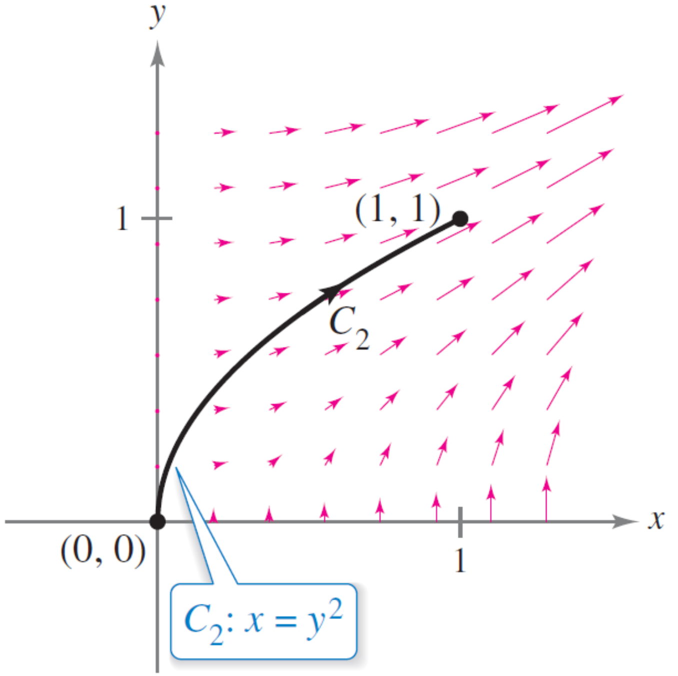
$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1=\int_0^1\mathbf{F}\big(x(t),y(t)\big)\cdot\mathbf{r}_1'(t)\,dt$$

$$=\int_0^1\left\langle\frac{1}{2}t^2,\frac{1}{4}t^2\right\rangle\cdot\langle 1,1\rangle\,dt=\int_0^1\frac{3}{4}t^2dt=\frac{1}{4}$$



LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x,y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.



2 $\quad C_2: \quad x = y^2 \qquad dx = 2y\,dy$

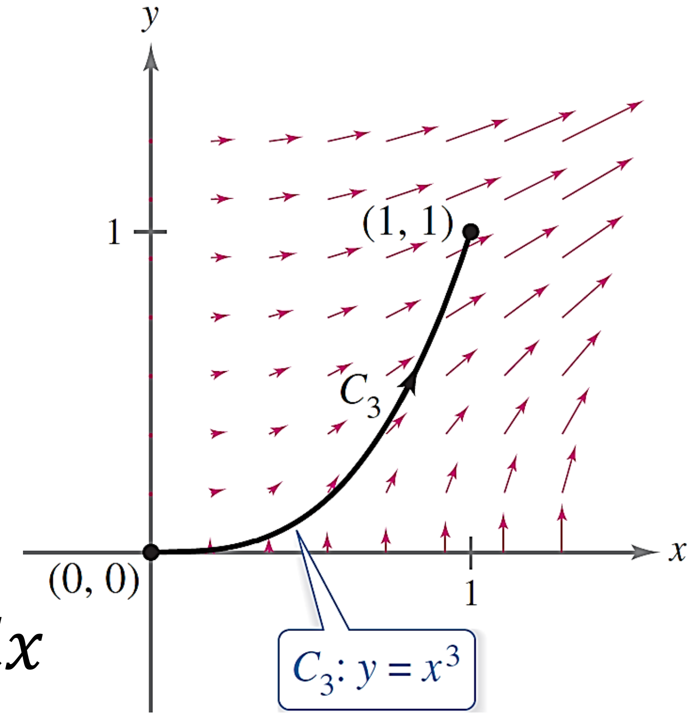
$$\int_{C_2} M\,dx + N\,dy = \int_0^1 \frac{1}{2}xy\,dx + \frac{1}{4}x^2\,dy$$

$$= \int_0^1 \frac{1}{2}(y^2)(y)(2y\,dy) + \frac{1}{4}(y^4)\,dy = \int_0^1 \frac{5}{4}y^4\,dy$$

$$= \frac{1}{4}$$

LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x,y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.



3 $\quad C_3: \quad y = x^3 \qquad dy = 3x^2dx$

$$\int_{C_3} Mdx + Ndy = \int_0^1 \frac{1}{2}xydx + \frac{1}{4}x^2dy$$

$$= \int_0^1 \frac{1}{2}(x)(x^3)(dx) + \frac{1}{4}(x^2)(3x^2dx) = \int_0^1 \frac{5}{4}x^4dx$$

$$= \frac{1}{4}$$

LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.

Note that the force field $\mathbf{F}(x, y) = \underbrace{\frac{1}{2}xy}_{M}\mathbf{i} + \underbrace{\frac{1}{4}x^2}_{N}\mathbf{j}$ is conservative.

$$\frac{\partial M}{\partial y} = \frac{x}{2} = \frac{\partial N}{\partial x}$$

So, the work done by the conservative vector field $\mathbf{F}$ is the same for each path.

LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x,y)=\frac{1}{2}xy\mathbf{i}+\frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.

Since the force field $\mathbf{F}(x,y)=\underbrace{\frac{1}{2}xy}_{M}\mathbf{i}+\underbrace{\frac{1}{4}x^2}_{N}\mathbf{j}$ is conservative, then

$$\mathbf{F}=\nabla f=\langle f_x,f_y\rangle$$
$$=\langle M,N\rangle$$

$$f_y=N=\frac{1}{4}x^2 \qquad f=\int\frac{1}{4}x^2dy=\frac{1}{4}x^2y+g(x)$$

$$f_x=M \qquad \frac{1}{2}xy+g'(x)=\frac{1}{2}xy \qquad g'(x)=0$$
$$g(x)=c$$

LINE INTEGRAL OF A CONSERVATIVE VECTOR FIELD

Example Find the work done by the force field $\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ on a particle that moves from $(0,0)$ to $(1,1)$ along each path, as shown in the figures.

Since the force field $\mathbf{F}(x, y) = \underbrace{\frac{1}{2}xy}_{M}\mathbf{i} + \underbrace{\frac{1}{4}x^2}_{N}\mathbf{j}$ is conservative, then

$$\mathbf{F} = \nabla f = \langle f_x, f_y \rangle$$

$$\boxed{f(x, y) = \frac{1}{4}x^2y + c}$$ Potential Function

Question: What is the value of $f(1, 1) - f(0, 0)$?

$$= \left[\frac{1}{4}(1^2)(1) + c\right] - \left[\frac{1}{4}(0^2)(0) + c\right] = \frac{1}{4}$$

FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad , \quad a \leq t \leq b$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

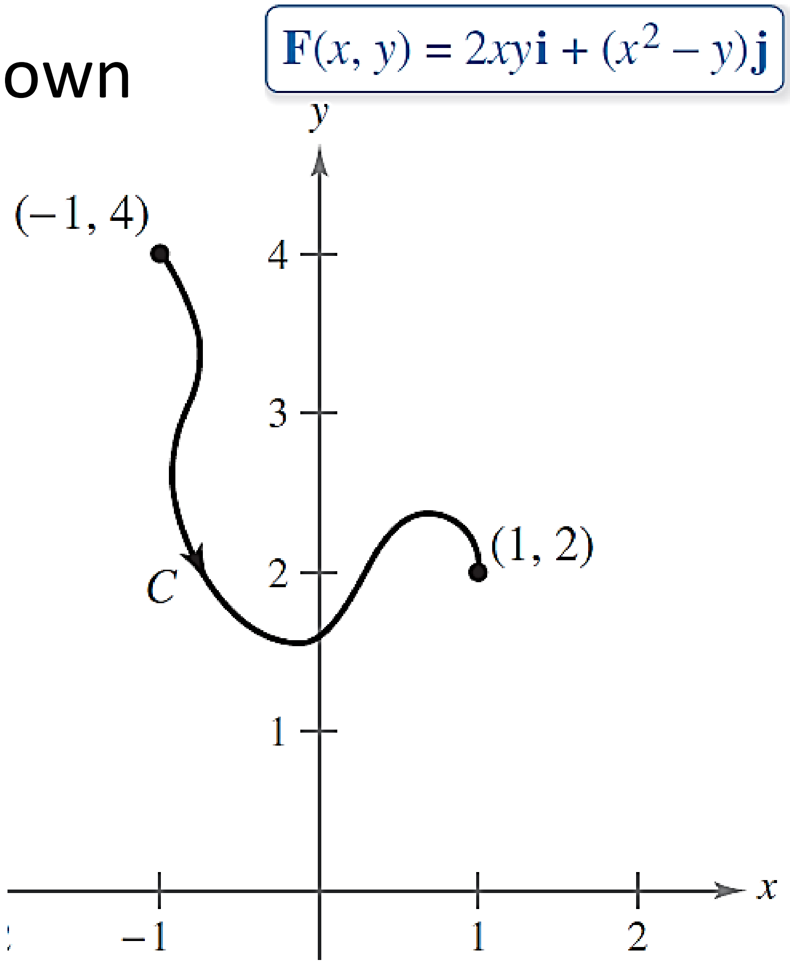
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\big(x(b), y(b)\big) - f\big(x(a), y(a)\big)$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

FUNDAMENTAL THEOREM OF LINE INTEGRALS

Example Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a piecewise smooth curve from $(-1,4)$ to $(1,2)$ as shown in the figure, and

$$\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$$



F is conservative

$$f(x,y) = x^2y - \frac{1}{2}y^2 + c$$

From a previous example in Section 15.1

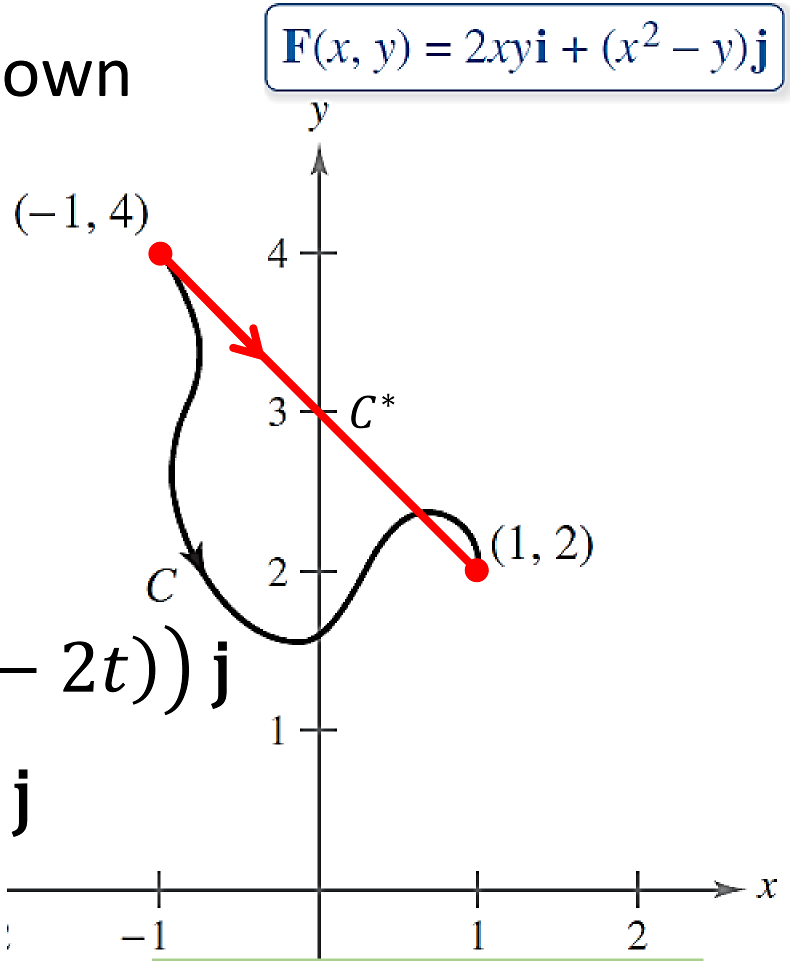
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(1,2) - f(-1,4)$$

$$= 0 - (-4) = 4$$

FUNDAMENTAL THEOREM OF LINE INTEGRALS

Example Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a piecewise smooth curve from $(-1,4)$ to $(1,2)$ as shown in the figure, and

$$\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$$



$$C^*: \quad \mathbf{r}(t) = (2t-1)\mathbf{i} + (4-2t)\mathbf{j} \quad t\in[0,1]$$

$$\mathbf{r}'(t) = 2\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{F}\big(x(t),y(t)\big) = 2(2t-1)(4-2t)\mathbf{i} + \big((2t-1)^2 - (4-2t)\big)\,\mathbf{j}$$

$$= (-8\,t^2 + 20\,t - 8)\mathbf{i} + (4\,t^2 - 2\,t - 3)\,\mathbf{j}$$

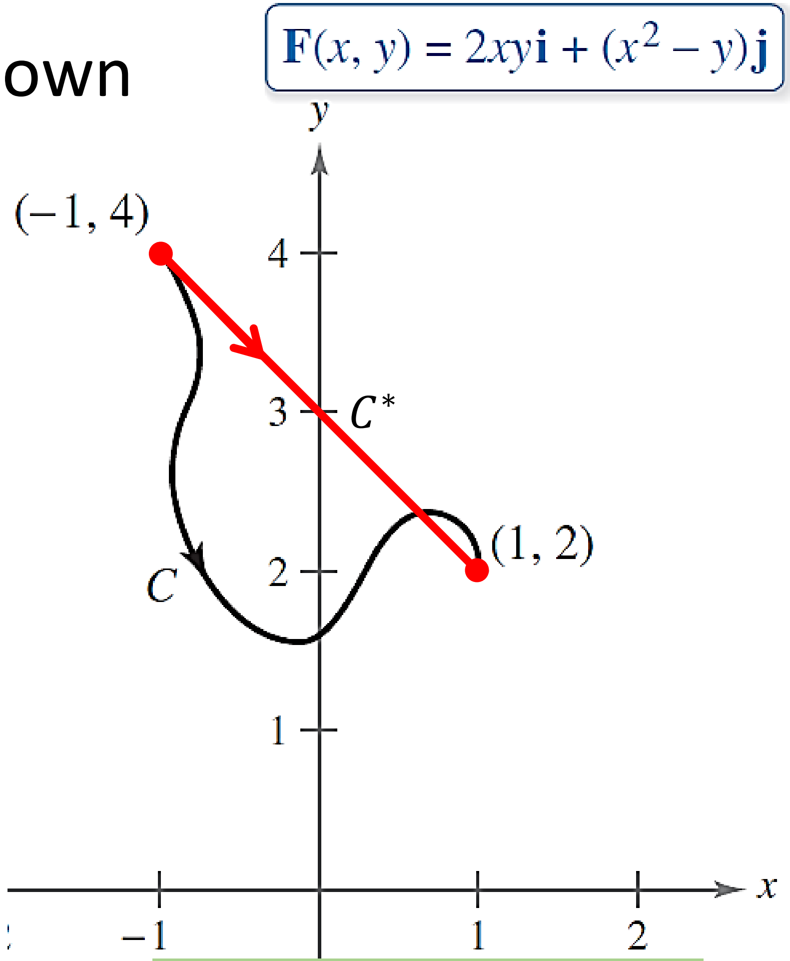
$$\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\mathbf{r}'(t)dt \quad = -24\,t^2 + 44\,t - 10$$

F is conservative

FUNDAMENTAL THEOREM OF LINE INTEGRALS

Example Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a piecewise smooth curve from $(-1,4)$ to $(1,2)$ as shown in the figure, and

$$\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$$



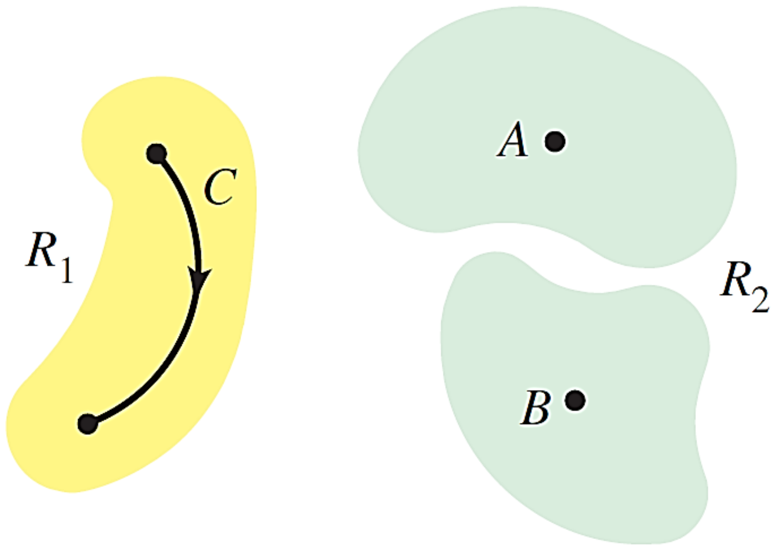
$$\mathbf{F} \cdot d\mathbf{r} = -24\,t^2 + 44\,t - 10$$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-24\,t^2 + 44\,t - 10)\,dt = 4$$

F is conservative

INDEPENDENCE OF PATH

A region in the plane (or in space) is **connected** when any two points in the region can be joined by a piecewise smooth curve lying entirely within the region.



R_1 is connected.

R_2 is not connected.

If $\mathbf{F}$ is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

EQUIVALENT STATEMENTS

$\vec{F}$ is conservative

$\vec{F}$ is gradient field ($\vec{F} = \nabla f$)

curl $\vec{F} = \vec{0}$

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path

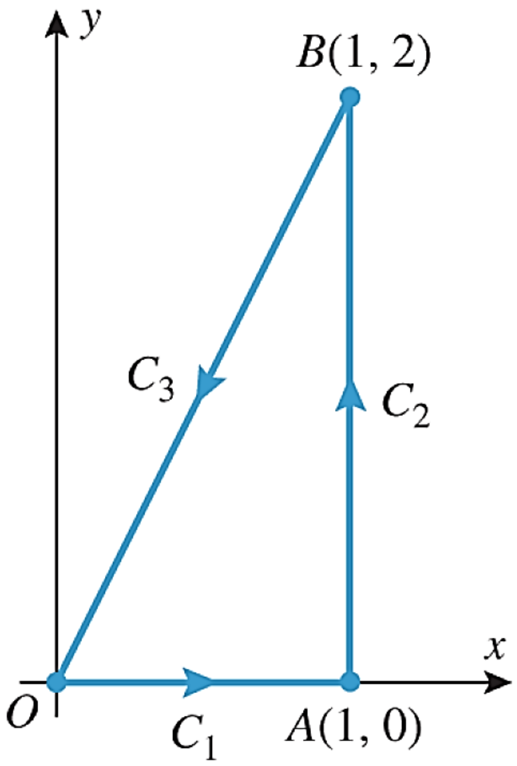
$\oint_C \vec{F} \cdot d\vec{r} = 0$

ORIENTATION AND PARAMETRIZATION OF A CURVE

Example We have seen that

$$\int_C x^2y\,dx + x\,dy = \frac{1}{2} \neq 0$$

where C is the triangular path shown in the figure.



$\therefore \mathbf{F} = x^2y\mathbf{i} + x\mathbf{j}$ is not conservative

is not independent of path

Closed Curve

Course: Calculus (4)

Chapter: [15]

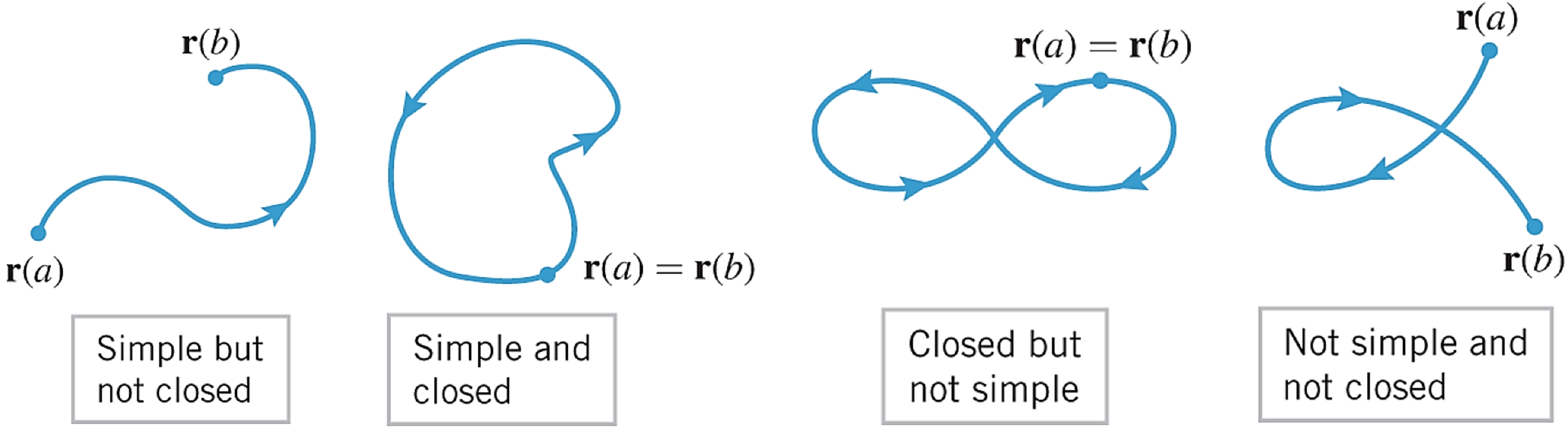
TOPICS IN VECTOR CALCULUS

Section: [15.4]

GREEN'S THEOREM

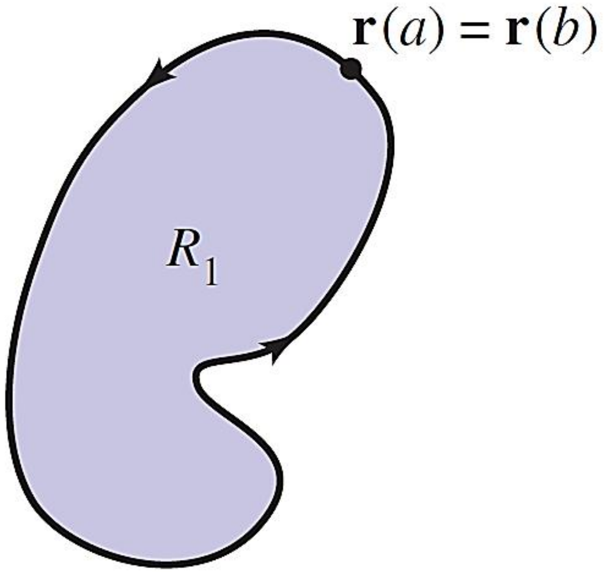
SIMPLE CURVES

- A curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where $a \le t \le b$, is **simple** when it does not cross itself between its endpoints.
- A simple parametric curve may or may not be **closed**.

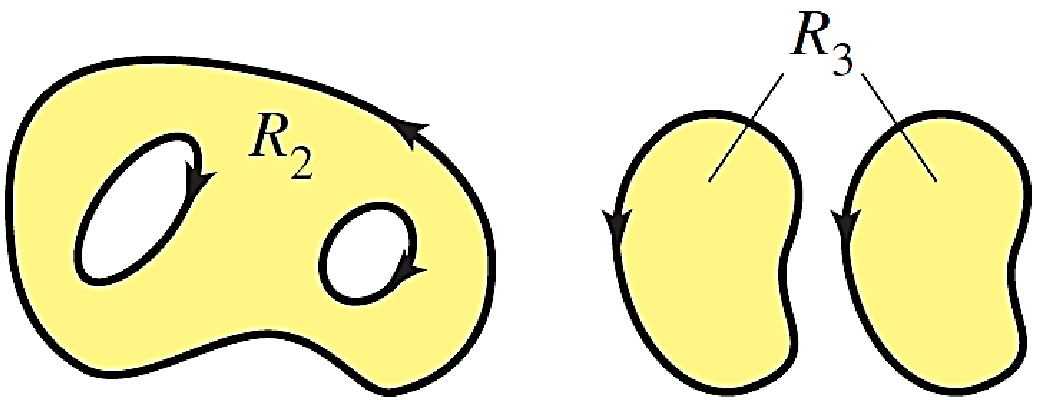


SIMPLY CONNECTED REGION

- A connected plane region R is **simply connected** when every simple closed curve in R encloses only points that are in R.
- Informally, a simply connected region cannot consist of *separate parts* or *holes*.



Simply connected



Not simply connected

GREEN'S THEOREM

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise. If M and N have continuous first partial derivatives in an open region containing R, then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

This theorem states that the value of a double integral over a simply connected plane region R is determined by the value of a line integral around the boundary of R.

GREEN'S THEOREM

An integral sign with a circle is sometimes used to indicate a line integral around a simple closed curve.

$$\oint$$

To indicate the orientation of the boundary, an arrow can be used.

$$\ointclockwise_C \qquad \text{Clockwise}$$

$$\ointctrclockwise_{\mathrm{C}} \qquad \text{Counterclockwise}$$

GREEN'S THEOREM

Example Use Green's Theorem to evaluate the line integral

$$\oint_C y^3 dx + (x^3 + 3xy^2) dy$$

where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ as shown in the figure.

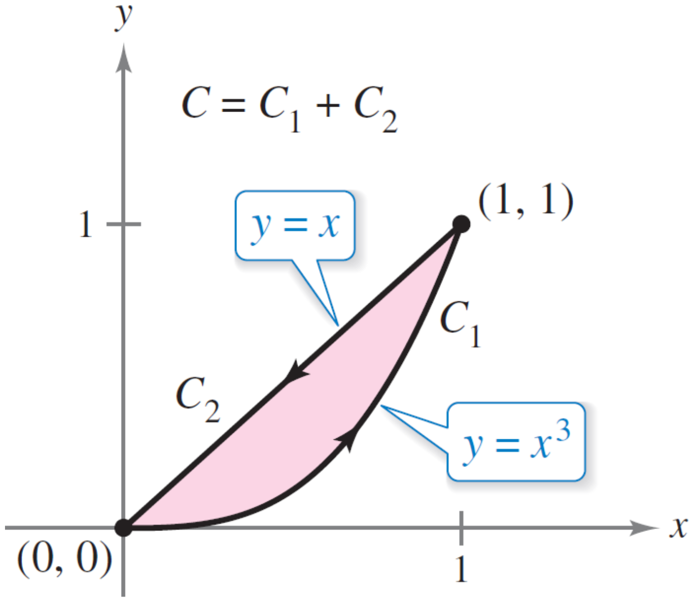
$$M = y^3$$

$$N = x^3 + 3xy^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\mathbf{F} = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$$

Not Conservative



GREEN'S THEOREM

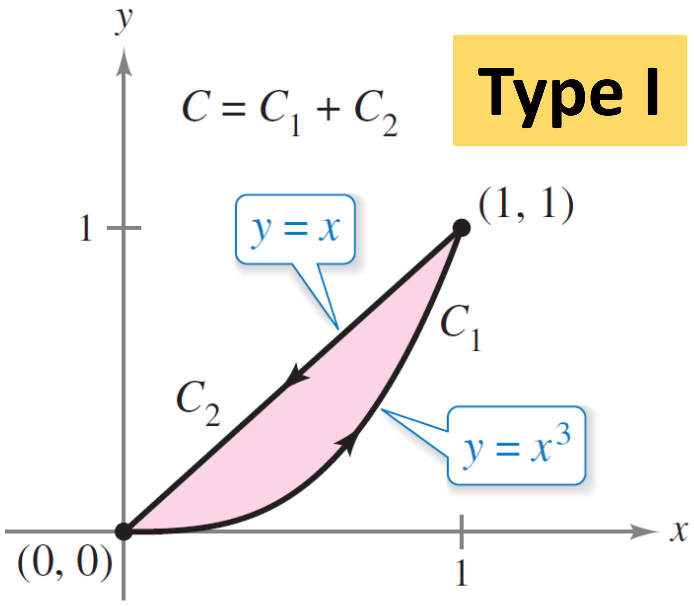
Example

$$\oint_C y^3 dx + (x^3 + 3xy^2) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (3x^2 + 3y^2 - 3y^2) dA$$

$$= \iint_R 3x^2 dA = \int_0^1 \int_{x^3}^{x} 3x^2 dy dx = \frac{1}{4}$$

Type I



$$M = y^3$$

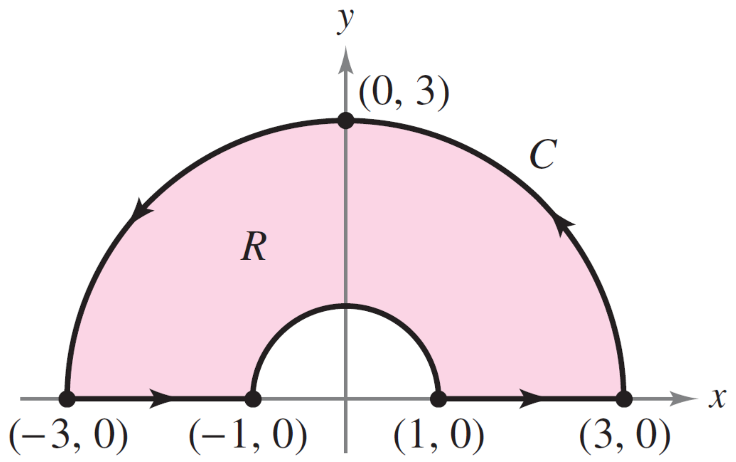
$$N = x^3 + 3xy^2$$

GREEN'S THEOREM

Example Evaluate

$$\oint_C (\tan^{-1} x + y^2)\,dx + (e^y - x^2)\,dy$$

where C is the path shown in the figure.



$$M = \tan^{-1} x + y^2$$

$$N = e^y - x^2$$

$$\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = -2x$$

$$\mathbf{F} = (\tan^{-1} x + y^2)\mathbf{i} + (e^y - x^2)\mathbf{j}$$

Not Conservative

GREEN'S THEOREM

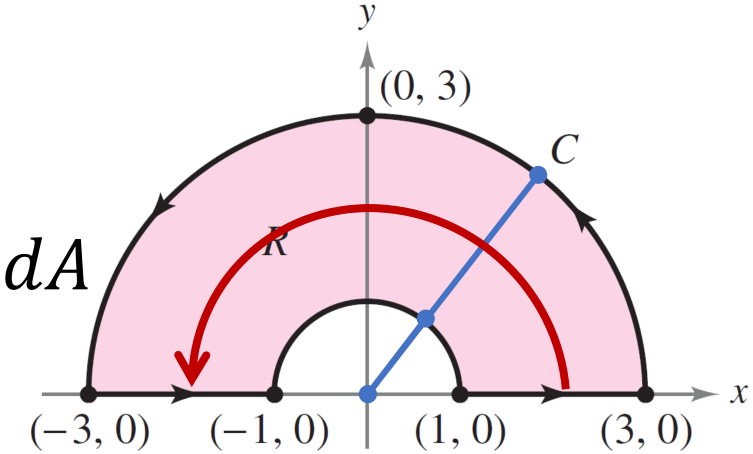
Example

$$\oint_C (\tan^{-1} x + y^2)dx + (e^y - x^2)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (-2x - 2y)dA$$

$$= \int_0^{\pi} \int_1^3 (-2r\cos\theta - 2r\sin\theta)r\,dr\,d\theta$$

$$= -\frac{104}{3}$$

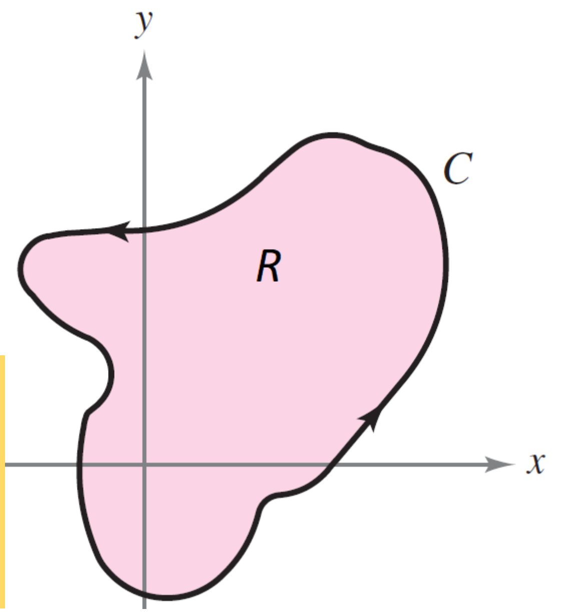


$$\frac{\partial N}{\partial x} = -2x \qquad \frac{\partial M}{\partial y} = 2y$$

Polar Coordinates

LINE INTEGRAL FOR AREA

Another application of Green's Theorem is in computing areas.



$$\text{Area of } R = \iint_R 1\, dA$$

If there is function M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$, then

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

For example, let $\frac{\partial M}{\partial y} = 0$

$M = c(x) = 0$

$$= \oint_C M dx + N dy$$

Then $\frac{\partial N}{\partial x} = 1$ $\quad N = x$

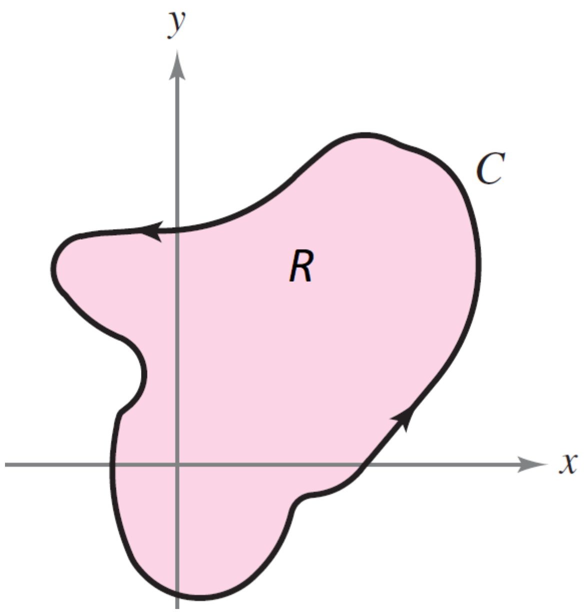
LINE INTEGRAL FOR AREA

If R is a plane region bounded by a piecewise smooth simple closed curve C oriented counterclockwise, then the area of R is given by

$$A = \oint_C x\,dy$$

$$= \oint_C -y\,dx$$

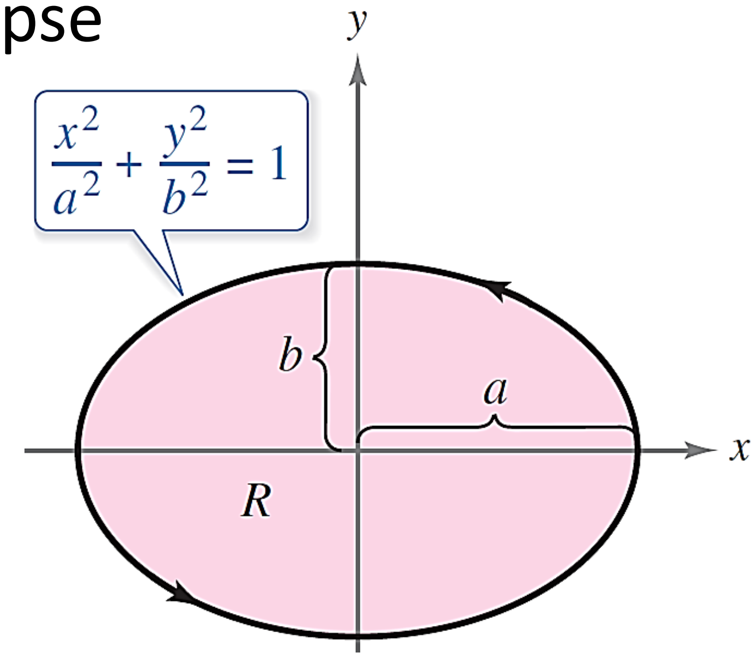
$$= \frac{1}{2}\oint_C x\,dy - y\,dx$$



LINE INTEGRAL FOR AREA

Example Use a line integral to find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



C: $\quad x = a\cos t \qquad y = b\sin t \qquad 0 \le t \le 2\pi$

$$A = \oint_C x\,dy \;=\; \int_0^{2\pi} (a\cos t)(b\cos t\;dt)$$

$$= \frac{ab}{2}\int_0^{2\pi} (1+\cos(2t))dt \;= ab\pi$$