Linear Programming
Lecture Notes for Math 373

Feras Awad

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Contents

1 Introduction to Linear Programming

A Linear Programming (LP) Problem is an **optimization** problem where the *goal* is to **maximize** or **minimize** a linear objective function, subject to a set of linear constraints. These constraints are typically expressed as linear equations or inequalities, and the variables are usually non-negative. LPPs are widely used in fields like economics, business, and engineering to allocate resources efficiently, minimize costs, or maximize profits.

1.1 What is a Linear Programming Problem?

Definition 1.1. A function $f\left(x_1, x_2, \cdots, x_n\right)$ of the variables $x_1, x_2, \cdots, x_n$ is a *linear function* if $f\left(x_1, x_2, \cdots, x_n\right) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ where $c_1, c_2, \cdots, c_n$ are constants.

Example 1.1. $f\left(x_1, x_2\right) = 2x_1 + x_2$ is linear function, while $f\left(x_1, x_2\right) = x_1^2 x_2$ is not linear.

Definition 1.2. For any linear function $f\left(x_1, x_2, \cdots, x_n\right)$ and any real number b, the inequalities $f\left(x_1, x_2, \cdots, x_n\right) \leq b$ and $f\left(x_1, x_2, \cdots, x_n\right) \geq b$ are *linear inequalities*.

Example 1.2. $2x_1 + 3x_2 \leq 3$ is linear inequality, while $x_1 x_2 + \sqrt{x_2} \geq 3$ is not linear inequality.

Definition 1.3. A *linear programming (LP) problem* is an optimization problem for which we do the following

1. We *maximize* (*profit*) or *minimize* (*cost*) a linear function (*objective function*) of the *decision variables*.
2. The decision variables must satisfy a set of linear equations or linear inequalities called *constraints*.
3. A *sign restriction* is associated with each variable. Each variable is either non-negative or is unrestricted in sign (urs).

max (or min) $z = f\left(x_1, x_2, \cdots, x_n\right)$
Subject to Constraints
Sign Restriction.

Example 1.3. Furnco manufactures desks and chairs. Each desk uses 4 units of wood, and each chair uses 8. A desk contributes \$40 to profit, and a chair contributes \$25. Marketing restrictions require that the number of chairs produced be at least twice the number of desks produced. If 20 units of wood are available, formulate an LP to maximize Furnco's profit.

Solution: Let x_1 be the number of desks produced, and x_2 be the number of chairs produced. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{max} & z = 40x_1 + 25x_2 \\ \textbf{s.t.} & 4x_1 + 8x_2 \leq 20 \\ & x_2 \geq 2x_1 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

The optimal solution of this problem is $x_1 = 1$, $x_2 = 2$, and $z = 90$. (We will see how we find these values later on.)

Note 1. [LP Assumptions]

1. **The Proportionality and Additivity Assumptions.**
   The fact that the objective function for an LP must be a linear function of the decision variables has two implications.

   (a) The objective function's contribution from each decision variable is proportional to its value. For instance, in example (1.3), making 5 desks contributes exactly five times the value of making one desk.

   (b) The contribution of any variable to the objective function is independent of other variables' values. For example, regardless of x_2, producing x_1 desks always adds $40x_1$ dollars to the objective function.

   Analogously, the fact that each LP constraint must be a linear inequality or linear equation has two implications.

   (a) The contribution of each variable to the lefthand side of each constraint is proportional to the value of the variable.

   (b) The contribution of a variable to the lefthand side of each constraint is independent of the values of the variable.

2. **The Divisibility Assumption.**
   The divisibility assumption allows decision variables to take fractional values, such as producing 1.5 desks or 1.63 chairs in example (1.3). However, since Frunco cannot produce fractional items, this assumption is not met in the Frunco problem. When variables must be non-negative integers, the problem is called an **integer programming problem**.

3. **The Certainty Assumption.**
   The certainty assumption is that each parameter (objective function coefficient, righthand side, and constraint coefficient) is known with certainty. If we were unsure of the exact amount of wood used by desks and chairs, the Certainty Assumption would be violated.

1.2 Modeling LP Problems

This section presents LP models in which the definition of the variables and the construction of the objective function and constraints are not straightforward.

Example 1.4. Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by \$10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Giapetto wants to maximize weekly profit (revenues - costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

Solution: Let x_1 = number of soldiers produced each week, x_2 = number of trains produced each week. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

Example 1.5. A psychologist uses two types of boxes with mice and rats. The amount of time (in minutes) that each mouse and each rat spends in each box per day is given in the table. Write an LP to maximize the number of mice and rats that can be used in this experiment?

	Time		Maximum Time
	Mice (x)	Rats (y)	Available per Day
Box A	10 min	20 min	800 min
Box B	20 min	10 min	640 min

Solution: Let x be the number of mice, and y be the number of rates. Then a correct formulation is

$$\begin{array}{ll} \textbf{max} & z = x + y \\ \textbf{s.t.} & 10x + 20y \leq 800 \\ & 20x + 10y \leq 640 \\ & x, y \geq 0 \end{array}$$

Example 1.6. A banquet hall offers two types of tables for rent: 6−person rectangular tables at a cost of $28 each and 10−person round tables at a cost of $52 each. Kathleen would like to rent the hall for a wedding banquet and needs tables for 250 people. The room can have a maximum of 35 tables and the hall only has 15 rectangular tables available. Formulate an LP to minimize the cost of each type of tables should be rented.

Solution: Let x_1 = number of rectangular tables, x_2 = number of round tables. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{min} & w = 28x_1 + 52x_2 \\ \textbf{s.t.} & 6x_1 + 10x_2 \geq 250 \\ & x_1 + x_2 \leq 35 \\ & x_1 \leq 15 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

Exercise 1.1.

1. Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs $4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs $1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily. Formulate an LP to minimize Leary Chemical's cost of daily demands.

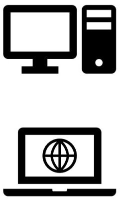
2. A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B. However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A and 4 lb per unit of B. The profit units for A and B are $20 and $50, respectively. Formulate an LP to maximize the company's profit.

3. A diet is to contain at least 400 units of vitamins, 500 units of minerals, and 1400 calories. Two foods are available: F1, which costs $0.05 per unit, and F2, which costs $0.03 per unit. A unit of food F1 contains 2 units of vitamins, 1 unit of minerals, and 4 calories; a unit of food F2 contains 1 unit of vitamins, 2 units of minerals, and 4 calories. Formulate

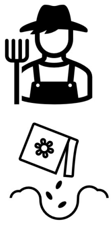
an LP to minimize the cost for a diet that consists of a mixture of these two foods and also meets the minimal nutrition requirements.

4. Katy needs at least 60 units of carbohydrates, 45 units of protein, and 30 units of fat each month. From each pound of food A, she receives 5 units of carbohydrates, 3 of protein, and 4 of fat. Food B contains 2 units of carbohydrates, 2 units of protein, and 1 unit of fat per pound. If food A costs $1.30 per pound and food B costs $0.80 per pound, formulate an LP to minimize the cost of pounds of each food should Katy buy each month.

5. An electronics firm manufactures two types of personal computers: a standard model and a portable model. The production of a standard computer requires a capital expenditure of $400 and 40 hours of labor. The production of a portable computer requires a capital expenditure of $250 and 30 hours of labor. The firm has $20000 capital and 2160 labor-hours available for production of standard and portable computers. If each standard computer contributes a profit of $320 and each portable model contributes a profit of $220, formulate an LP to maximize the firm's profit.



6. A farmer owns a 100-acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs $40, $20, and $30 per acre, respectively. A maximum of $3200 can be spent on seed. Crops A, B, and C require one, two, and one work days per acre, respectively, and there are a maximum of 160 work days available. If the farmer can make a profit of $100 per acre on crop A, $300 per acre on crop B, and $200 per acre on crop C, how many acres of each crop should be planted to maximize profit?



7. Farmer Jones bakes two types of cake (chocolate and vanilla) to supplement his income. Each chocolate cake can be sold for $1, and each vanilla cake can be sold for 0.5$. Each chocolate cake requires 20 minutes of baking time and uses 4 eggs. Each vanilla cake requires 40 minutes of baking time and uses 1 egg. Eight hours of baking time and 30 eggs are available. Let x_1 be the number of chocolate cakes baked, and x_2 be the number of vanilla cakes baked. Formulate an LP to maximize Farmer Jones's revenue.

1.3 Geometric Preliminaries and Solutions

Any LP with only two variables can be solved graphically. We always label the variables x_1 and x_2 and the coordinate axes the x_1 and x_2 axes.

1.3.1 Half–Spaces, Hyperplanes, and Convex Sets

Definition 1.4. We define the *Euclidean plane* $\mathbb{R}^n$ to be the set of all n–tuples of real numbers; that is

$$\mathbb{R}^n = \left\{ (x_1, x_2, \cdots, x_n) \,\middle|\, x_i \in \mathbb{R} \text{ for } i = 1, 2, \cdots, n \right\}$$

For example, $\mathbb{R}^2 = \left\{ (x_1, x_2) \,\middle|\, x_1 \text{ and } x_2 \text{ are reals} \right\}$. Geometrically, we represent $\mathbb{R}^2$ as in Figure 1.

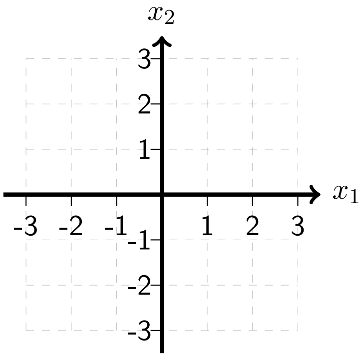


Figure 1 Coordinates Plane, $\mathbb{R}^2$

The graph in $\mathbb{R}^2$ of an equation of the form $a_1x_1 + a_2x_2 = c$ (where a_1, a_2, c are constants) is a straight line. For example, the graph in $\mathbb{R}^2$ of the equation $2x_1 - 3x_2 = 6$ is the line indicated in Figure 2.

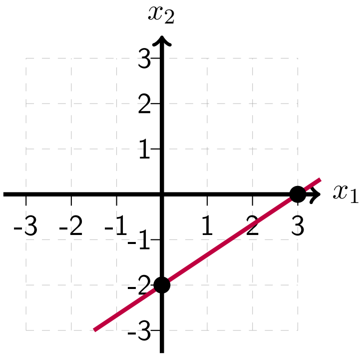


Figure 2 The line $2x_1 - 3x_2 = 6$

The graph in $\mathbb{R}^2$ of the inequality $a_1x_1 + a_2x_2 \leq c$ or $a_1x_1 + a_2x_2 \geq c$ is the set of all points in $\mathbb{R}^2$ lying on the line $a_1x_1 + a_2x_2 = c$ together with all points lying to one side of this line. For example, the shaded region in Figure 3 is the graph of the inequality $2x_1 - 3x_2 \leq 6$. To determine on which side of the line where the region of the inequality $2x_1 - 3x_2 \leq 6$ lies, consider a point, say $(0, 0)$, not lying on the line but satisfying the inequality; the side of the line containing this point is the one corresponding to the inequality.

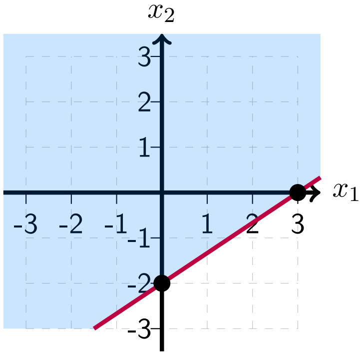


Figure 3 The inequality $2x_1 - 3x_2 \leq 6$

Definition 1.5. A *half–space* in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ satisfying an inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c$$

or an inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$$

where at least one of the constants $a_1, a_2, \cdots, a_n$ is nonzero.

Definition 1.6. A *hyperplane* in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ satisfying an equality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where at least one of the constants $a_1, a_2, \cdots, a_n$ is nonzero.

For example, the set of points in $\mathbb{R}^5$ satisfying

$$3x_1 + \frac{1}{2}x_2 - x_3 + x_4 + \frac{2}{3}x_5 = -9$$

is a hyperplane in $\mathbb{R}^5$, and the set of points in $\mathbb{R}^5$ satisfying

$$3x_1 + \frac{1}{2}x_2 - x_3 + x_4 + \frac{2}{3}x_5 \geq -9$$

is a half-space in $\mathbb{R}^5$.

Definition 1.7. A subset K of $\mathbb{R}^n$ is *convex* if K is empty, or K is a single point, or if for each two distinct points $\mathbf{p}$ and $\mathbf{q}$ in K, the line segment connecting $\mathbf{p}$ and $\mathbf{q}$ lies entirely in K.

Example 1.7. The sets in Figure 4 are convex.

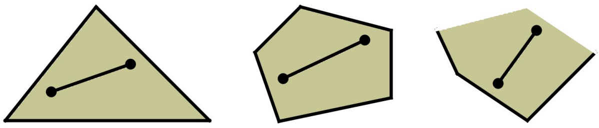


Figure 4 Convex sets

Example 1.8. The sets in Figure 5 are not convex.

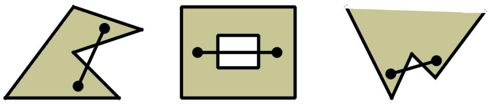


Figure 5 Not convex sets

Definition 1.8. If $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ and $\mathbf{q} = (q_1, q_2, \cdots, q_n)$ are points in $\mathbb{R}^n$, then the *line segment* joining $\mathbf{p}$ and $\mathbf{q}$ consists of all points of the form

$$(1-t)\mathbf{p} + t\mathbf{q}\,; \qquad 0 \leq t \leq 1$$

where

$$\begin{aligned}(1-t)\mathbf{p} + t\mathbf{q} &= (1-t)\,(p_1, p_2, \cdots, p_n) + t\,(q_1, q_2, \cdots, q_n) \\ &= \Big[(1-t)p_1 + tq_1, \cdots, (1-t)p_n + tq_n\Big]\end{aligned}$$

Observe that if $t = 0$, then $(1-t)\mathbf{p} + t\mathbf{q} = \mathbf{p}$, and if $t = 1$, then $(1-t)\mathbf{p} + t\mathbf{q} = \mathbf{q}$.

Example 1.9. The line segment in $\mathbb{R}^2$ joining the points $\mathbf{p} = (3, 6)$ and $\mathbf{q} = (-4, 5)$ is the set of points

$$\begin{aligned}(1-t)\mathbf{p} + t\mathbf{q} &= (1-t)\,(3,6) + t\,(-4,5)\\ &= \Big[3(1-t) - 4t, 6(1-t) + 5t\Big] = \big(3 - 7t, 6 - t\big)\ ;\ 0 \le t \le 1\end{aligned}$$

Theorem 1.1. *A half-space H in $\mathbb{R}^n$ that is defined either by the inequality*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le c$$

or the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \ge c$$

is convex.

Proof. We establish this result for the half-space H defined by the inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le c \tag{1.1}$$

A similar argument hold for half-spaces defined by $a_1x_1 + a_2x_2 + \cdots + a_nx_n \ge c$. Suppose the points $\mathbf{p} = (p_1, \cdots, p_n)$ and $\mathbf{q} = (q_1, \cdots, q_n)$ lie in H; that is, these points satisfy inequality (1.1), so we have

$$\begin{aligned}a_1p_1 + a_2p_2 + \cdots + a_np_n \le c\\ a_1q_1 + a_2q_2 + \cdots + a_nq_n \le c\end{aligned}$$

To show that the line segment connecting these two points lies entirely in H, it suffices to show that for each $t \in \big[0, 1\big]$, the point

$$(1-t)\mathbf{p} + t\mathbf{q} = \Big[(1-t)p_1 + tq_1, \cdots, (1-t)p_n + tq_n\Big]$$

also satisfies inequality (1.1). To show this, we have

$$\begin{aligned}&a_1\left[(1-t)p_1 + tq_1\right] + a_2\left[(1-t)p_2 + tq_2\right] + \cdots + a_n\left[(1-t)p_n + tq_n\right]\\ &\quad = (1-t)\left(a_1p_1 + a_2p_2 + \cdots + a_np_n\right) + t\left(a_1q_1 + a_2q_2 + \cdots + a_nq_n\right)\\ &\quad \le (1-t)c + tc = c\end{aligned}$$

and this concludes the proof. ■

Theorem 1.2. *If $K_1, K_2, \cdots, K_r$ are convex subsets of $\mathbb{R}^n$, then the intersection of these sets, $K = K_1 \cap K_2 \cap \cdots \cap K_r$ is also convex.*

Proof. If K is empty or consists of a single point, then K is convex by definition (1.7). Suppose then that K consists of more than one point, and let $\mathbf{p}$ and $\mathbf{q}$ be any two distinct points in K. Since $\mathbf{p}$ and $\mathbf{q}$ are in each convex set K_i; $1 \leq i \leq r$, the line segment L connecting $\mathbf{p}$ and $\mathbf{q}$ also lies entirely in each K_i. Therefore, L lies in the intersection K of these sets, and we conclude that K is convex. ■

Theorem 1.3. *A hyperplane M in $\mathbb{R}^n$ defined by*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

is convex

Proof. M is the intersection of the convex half-spaces

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c$$

and

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$$

By Theorem (1.2), this intersection is convex. ■

Definition 1.9. A point $\mathbf{q}$ is a *corner point* (or an *extreme point*) of a convex set K if $\mathbf{q}$ is not an interior point of any line segment contained in K.

Example 1.10. The points $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{q_4}, \mathbf{q_5}$ are corner points of the convex set in Figure 6.

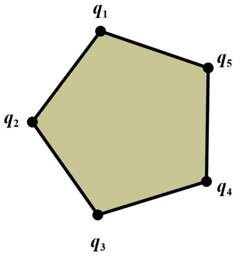


Figure 6

Exercise 1.2.

1. Draw the graph in $\mathbb{R}^2$ of the following half-spaces.

   (a) $-2x_1 + 4x_2 \geq 12$.

   (b) $x_2 \leq 2x_1$.

   (c) $x_1 \geq 4$.

   (d) $-3x_2 \leq 9$

2. Which of the following expressions define hyperplanes, half-spaces, or neither?

   (a) $2x_1 + 3x_2 = x_2 - x_4 + 3$.

   (b) $x_1 - 3x_4 \geq 3x_2 + x_3$.

   (c) $x_1 x_2 \leq 1$.

   (d) $x_1 = 6 + \dfrac{2}{x_2}$.

   (e) $2.5x_1 - 3.2x_2 = 10$.

   (f) $x_1 + x_2^3 \geq 9$.

3. Which of the following sets in Figure 7 are convex?

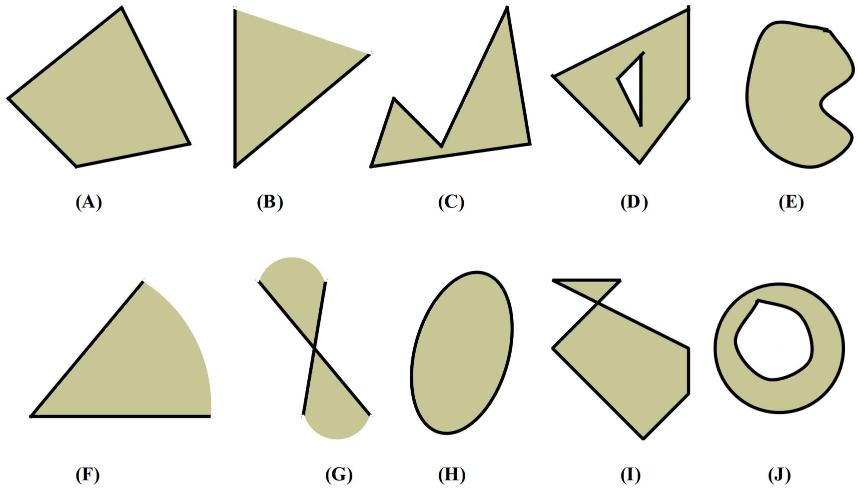


Figure 7

4. Let $\mathbf{p} = (1, 3, 2)$ and $\mathbf{q} = (2, 4, -1)$ be two points in $\mathbb{R}^3$.

   (a) Find the set of points that lie on the line segment joining the points $\mathbf{p}$ and $\mathbf{q}$.

   (b) Show that the point $(1.5, 3.5, 0.5)$ lies on the line segment joining the points $\mathbf{p}$ and $\mathbf{q}$.

1.3.2 The Graphical Solution of Two−Variable LP Problems

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution. For defining these concepts, we use the term *point* to mean a specification of the value for each decision variable.

Definition 1.10. The *feasible region* for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions. Any point that is not in LP's feasible region is said to be *infeasible point*.

The shaded area in Figure 8 indicates the feasible region of the LP in example (1.3). Note that each of the constraints in the LP defines a half-space. The feasible set consists of all points in the intersection of these half-spaces. Observe that the feasible region in Figure 8 is convex. Note that the points $(0, 0)$, $(0.5, 1.5)$, and $(1, 2)$ are all in the feasible region, while $(1, 1)$ is infeasible, because it does not satisfy the second constraint.

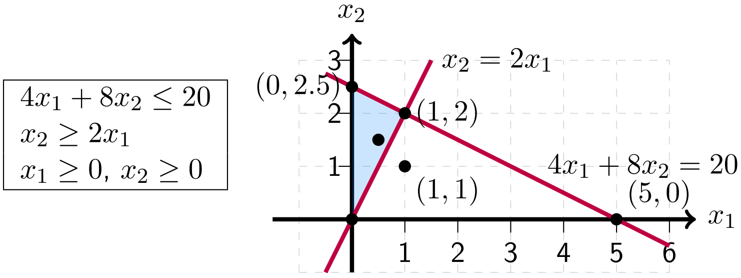


Figure 8 The feasible region of example (1.3)

Theorem 1.4. *The feasible region in $\mathbb{R}^n$ corresponding to any number of constraints of the types below is convex.*

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &\leq b \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &= b \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &\geq b \\ x_1, x_2, \cdots, x_n &\geq 0 \end{aligned}$$

Proof. The inequality constraints define half-spaces, and the equality constraints define hyperplanes. By Theorems 1.1 and 1.3 these half-spaces and hyperplanes are convex sets. Since the feasible region is the intersection of these convex sets, it follows from Theorem 1.2 that the feasible region is convex. ■

Definition 1.11. For a maximization (minimization) problem, an *optimal solution* to an LP is a point in the feasible region with the largest (smallest) objective function value.

The goal of any LP problem is to find the optimum, the best feasible solution that maximizes the total profit or minimizes the cost. Having identified the feasible region for the Furnco problem in example (1.3) as shown in Figure 8, we now search for the optimal solution, which will be the point in the feasible region with the largest value of $z = 40x_1 + 25x_2$.

- To find the optimal solution, we need to graph a line on which all points have the same $z-$value. In a max problem, such a line is called an *isoprofit* line.
- To draw an isoprofit line, choose any point in the feasible region and calculate its $z-$value. Let us choose $(0, 1)$. For $(0, 1)$, $z = 40(0) + 25(1) = 25$. Thus, $(0, 1)$ lies on the isoprofit line $z = 40x_1 + 25x_2 = 25$.
- Because all isoprofit lines are of the form $40x_1 + 25x_2 =$ constant, all isoprofit lines have the same slope. This means that once we have drawn one isoprofit line, we can find all other isoprofit lines by moving parallel to the isoprofit line we have drawn in a direction that increases z.
- After a point, the isoprofit lines will no longer intersect the feasible region. The last isoprofit line intersecting (touching) the feasible region defines the largest $z-$value of any point in the feasible region and indicates the optimal solution to the LP.
- In our problem, the objective function $z = 40x_1 + 25x_2$ will increase if we move in a direction for which both x_1 and x_2 increase. Thus, we construct additional isoprofit lines by moving parallel to $40x_1+25x_2 = 25$ in a northeast direction (upward and to the right), as shown in Figure 9.

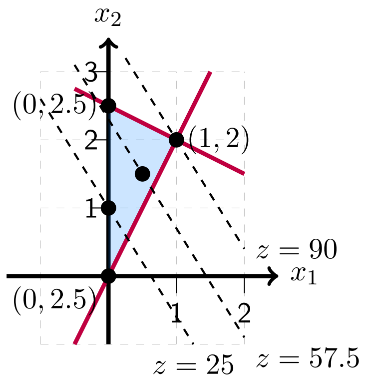


Figure 9 Isoprofit lines

- From Figure 9, we see that the isoprofit line passing through point $(1, 2)$ is the last isoprofit line to intersect the feasible region. Thus, $(1, 2)$ is the point in the feasible region with the largest $z-$value and is therefore the optimal solution to the Furnco problem. Thus, the optimal value of z is $z = 40(1) + 25(2) = 90$.

Note 2. In a minimization problem, to find the optimal solution, we need to graph a line on which all points have the same $w-$value, such a line is called an *isocost* line. Once we have drawn one isocost line, we can find all other isocost lines by moving parallel to the isocost line we have drawn first in a direction that decreases w. After a point, the isocost lines will no longer intersect the feasible region. The last isocost line intersecting the feasible region defines the smallest $w-$value of any point in the feasible region and indicates the optimal solution to the LP.

Example 1.11. Graphically solve the following LP problem.

$$\begin{array}{ll}
\textbf{min} & w = -4x_1 + 7x_2 \\
\textbf{s.t.} & x_1 + x_2 \geq 3 \\
 & -x_1 + 2x_2 \leq 6 \\
 & 2x_1 + x_2 \leq 8 \\
 & x_1 \geq 0,\ x_2 \geq 0
\end{array}$$

Solution: From Figure 10,The isocost lines minimizes the value of the objective function by decreasing w in the southeast direction (down-right). The optimum solution is the intersection of the two lines $2x_1 + x_2 = 8$ and $x_2 = 0$, which yields $x_1 = 4$ and $x_2 = 0$. The minimum value of w is $w = -4(4)+7(0) = -16$.

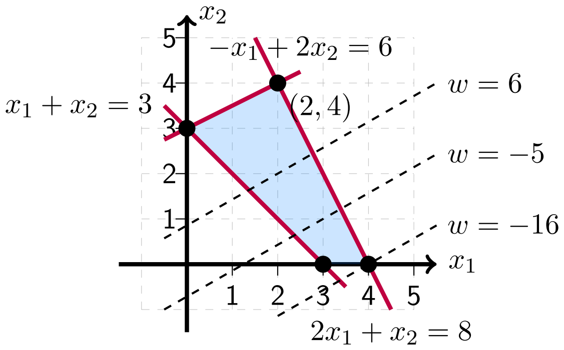


Figure 10

The Corner-Point Theorem In practice, a typical LP may include hundreds or even thousands of variables and constraints. Of what good then is the study of a two-variable LP? The answer is that the graphical solution provides one of the most important key result in linear programming: "*The optimum solution of an LP, when it exists, is always associated with a corner point of the solution space*", thus limiting the search for the optimum from an infinite number of feasible points to a finite number of corner points. This powerful result is the basis for the development of the general algebraic *simplex method* presented in Chapter 2.

Theorem 1.5. [Corner Point Theorem]
Consider the LP problem

Maximize (or Minimize) $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to a system of inequalities of the types

$$\begin{aligned} c_1x_1 + c_2x_2 + \cdots + c_nx_n &\leq b \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n &= b \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n &\geq b \\ x_1, x_2, \cdots, x_n &\geq 0 \end{aligned}$$

1. *If the feasible region is bounded, then the optimal solution is attained at a corner point of this feasible region.*
2. *If the feasible region is unbounded, then an optimal solution may not exist; however, if an optimal solution exists, it is attained at a corner point of this feasible region.*

Proof. The proof of this theorem can be found in: **Jan Van Tiel, Convex Analysis, New York: Wiley, 1984.** ■

Note 3. If the feasible region of an LP is bounded, then an optimal solution exists and we use The Corner Point Theorem (1.5) to find this optimal solution. In the case that the feasible region is unbounded, the isoprofit (isocost) line is used to determine whether the LP has an optimal solution or not.

Example 1.12. Graphically solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

Once the optimal solution to an LP has been found, it is useful to classify each constraint as being a binding constraint or a nonbinding constraint.

Solution: From Figure 11, the feasible region is bounded, so we use Theorem (1.5) as follows:

Corner	z	
$(0,0)$	0	
$(0,80)$	160	
$(20,60)$	180	**Optimal**
$(40,20)$	160	
$(40,0)$	120	

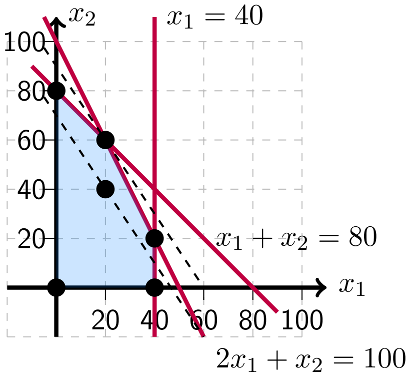


Figure 11

Definition 1.12. A constraint is *binding* if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint. Otherwise, it is called *nonbinding*.

For instance, in example (1.12), the first two constraints are binding, while the third one is nonbinding. While in example (1.11) the third constraint is binding and the other two constraints are nonbinding.

Definition 1.13. A constraint is said to be *redundant* if its removal from the model leaves the feasible solution space unchanged.

For example, the feasible region of the following constraints has a redundant constraint as shown in Figure 12

Constraint [1]: $2x_1 + x_2 \leq 6$
Constraint [2]: $x_1 + 3x_2 \leq 9$
Constraint [3]: $x_1 + x_2 \leq 5$
Sign Restriction: $x_1, x_2 \geq 0$

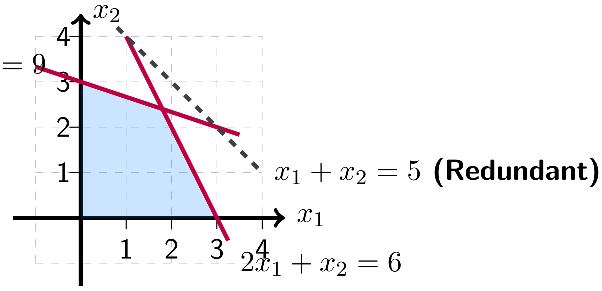


Figure 12

Note that the third constraint is the redundant constraint since its removal from the region will leave the feasible region unchanged.

Note 4. [**Alternative Optima**] An LP problem has infinitely many optimal solutions when the objective function is parallel to a nonredundant binding constraint. As an isoprofit (or isocost) line exits the feasible region, it intersects an entire segment of the binding constraint. If two points are optimal, any point on the segment between them is also optimal. In such cases, a secondary criterion, like *goal programming*, helps select among optimal solutions. The next example highlights the practical importance of this.

Example 1.13. Graphically solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 4x_1 + x_2 \\ \textbf{s.t.} & 8x_1 + 2x_2 \leq 16 \\ & 5x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: The feasible region for this LP is the shaded bounded region in Figure 13. So we use Theorem (1.5) as follows:

Corner	z	
$(0,0)$	0	
$(0,6)$	6	
$(4/3, 8/3)$	8	**Optimal**
$(2,0)$	8	**Optimal**

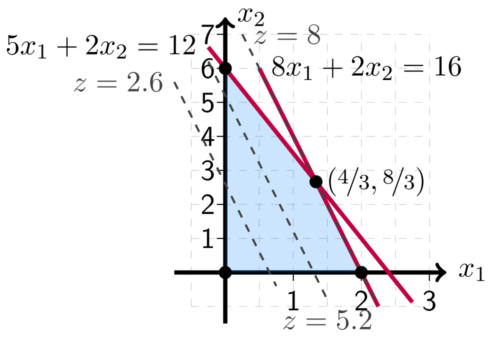


Figure 13

From the table above, the optimal solution occurs at the two corner points $\left(\frac{4}{3}, \frac{8}{3}\right)$ and $(2,0)$. This means that any point on line segment joining these corner points optimal. These points are given by

$$t\left(\frac{4}{3}, \frac{8}{3}\right) + (1-t)(2,0) = \left(2 - \frac{2t}{3}, \frac{8t}{3}\right) \ ; \ 0 \leq t \leq 1$$

Note 5. [**Infeasible LP**] An LP is infeasible if its feasible region is empty, meaning no solution exists. Since an optimal solution requires a feasible region, an infeasible LP has none.

Example 1.14. The following LP problem has no feasible solution as shown in Figure 14.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 3x_1 + 2x_2 \leq 120 \\ & x_1 + x_2 \leq 50 \\ & x_1 \geq 30 \\ & x_2 \geq 20 \\ & x_1, x_2 \geq 0 \end{array}$$

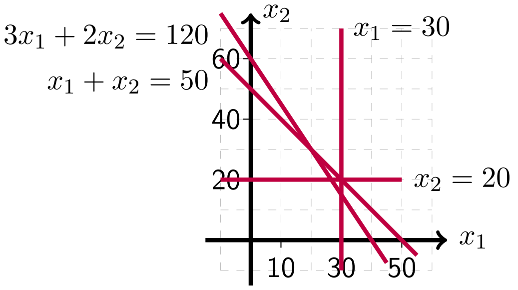


Figure 14

Note 6. [**Unbounded LP**] An LP is unbounded if its feasible region allows arbitrarily large (for maximization) or arbitrarily small (for minimization) z-values. This suggests an error in formulation or input. Graphically, a maximization problem is unbounded if moving parallel to the isoprofit line in the increasing z-direction never leaves the feasible region. For minimization, unboundedness occurs if moving in the decreasing z-direction never exits the feasible region.

Example 1.15. Graphically solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 - x_2 \\ \textbf{s.t.} & x_1 - x_2 \leq 1 \\ & 2x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: The feasible region is the (shaded) unbounded region in Figure 15. To find the optimal solution, we draw the isoprofit line passing through $(2, 5)$. This isoprofit line has $z = 2(2) - (5) = -1$. The direction of increasing z is to the southeast (this makes x_1 larger and x_2 smaller). Moving parallel to $z = 2x_1 - x_2$ in a southeast direction, we see that any isoprofit line we draw will intersect the feasible region. (This is because any isoprofit line is steeper than the line $x_1 - x_2 = 1$.) Thus, there are points in the feasible region that have arbitrarily large z-values.

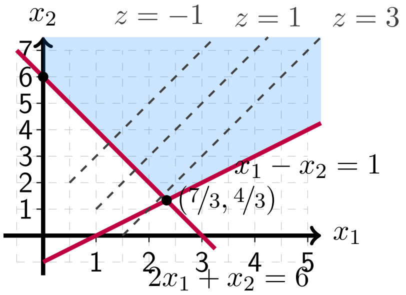


Figure 15

Example 1.16. Graphically solve the following LP problem.

$$\begin{array}{ll} \textbf{min} & w = x_1 + 2x_2 \\ \textbf{s.t.} & x_1 + x_2 \geq 6 \\ & 2x_1 - x_2 \geq 0 \\ & x_1 - 2x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: Although the feasible region of the LP is unbounded, but it has optimal solution as shown in Figure 16.

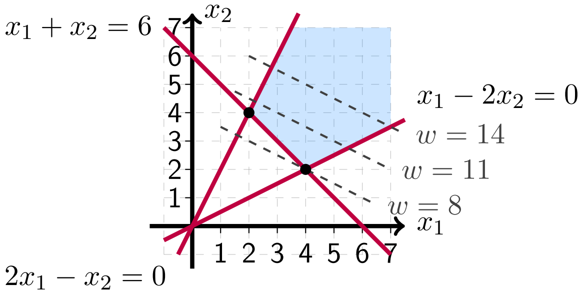


Figure 16

Example 1.17. Graphically solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 5x_1 + 6x_2 \\ \textbf{s.t.} & x_1 - 2x_2 \geq 1 \\ & -2x_1 + x_2 \geq 1 \\ & x_1, x_2 \textit{ urs} \end{array}$$

Solution: The two variables x_1 and x_2 are unrestricted in sign means that both can be positive, negative, or zero. The feasible region is the shaded region in Figure 17. To find the optimal solution, we draw the isoprofit line passing through $(-3, 3)$. This isoprofit line has $z = 5(-3) + 6(-3) = -33$. The direction of increasing z is to the northeast. Moving parallel to $z = 5x_1 + 6x_2$ in a northeast direction, we see that the last isoprofit line we draw will touch the feasible region at the point $(-1, -1)$. Thus, the optimal $z-$value is $z = 5(-1) + 6(-1) = -11$.

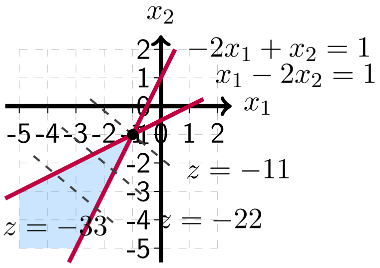


Figure 17

Exercise 1.3.

1. Match the solution region of each system of linear inequalities with one of the four regions shown in Figure 18.

   (a) $x + 2y \leq 8$
   $3x - 2y \geq 0$

   (b) $x + 2y \geq 8$
   $3x - 2y \leq 0$

   (c) $x + 2y \geq 8$
   $3x - 2y \geq 0$

   (d) $x + 2y \leq 8$
   $3x - 2y \leq 0$

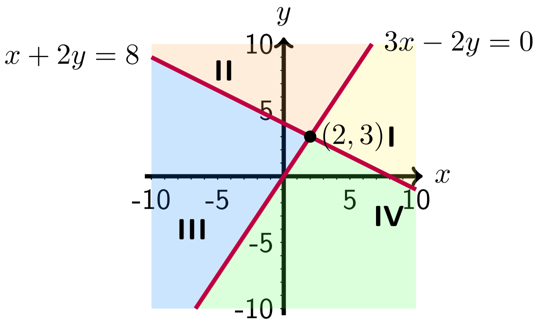


Figure 18

2. Find the maximum value of each objective function over the feasible region shown in Figure 19.

   (a) $z = x + y$

   (b) $z = 4x + y$

   (c) $z = 3x + 7y$

   (d) $z = 9x + 3y$

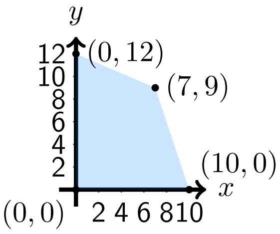


Figure 19

3. Find the minimum value of each objective function over the feasible region shown in Figure 20.

   (a) $w = 7x + 4y$

   (b) $w = 7x + 9y$

   (c) $w = 3x + 8y$

   (d) $w = 5x + 4y$

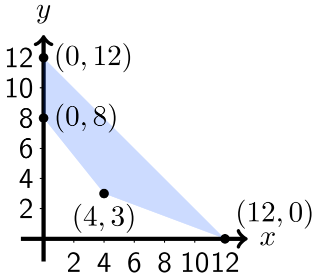


Figure 20

4. The corner points for the bounded feasible region determined by the system of linear inequalities

$$\begin{aligned} x + 2y &\leq 10 \\ 3x + y &\leq 15 \\ x, y &\geq 0 \end{aligned}$$

are $O = (0,0)$, $A = (0,5)$, $B = (4,3)$, and $C = (5,0)$ as shown in Figure 21. If $P = ax + by$ and $a, b > 0$, determine conditions on a and b that will ensure that the maximum value of P occurs

(a) only at A

(b) only at B

(c) only at C

(d) at both A and B

(e) at both B and C

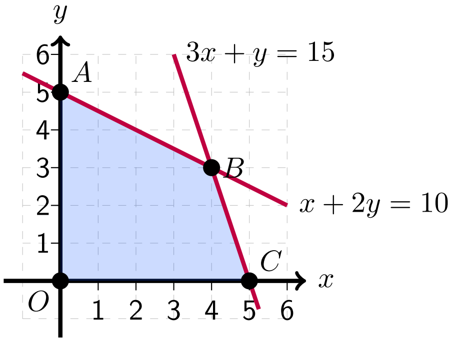


Figure 21

5. Identify the direction of increase in z in each of the following cases:
   (a) Maximize $z = x_1 - x_2$.
   (b) Maximize $z = -8x_1 - 3x_2$.
   (c) Maximize $z = -x_1 + 3x_2$.
6. Identify the direction of decrease in w in each of the following cases:
   (a) Minimize $w = 4x_1 - 2x_2$.
   (b) Minimize $w = -6x_1 + 2x_2$.
7. Determine the solution space graphically for the following inequalities. Which constraints are redundant? Reduce the system to the smallest number of constraints that will define the same solution space.

$$\begin{aligned} x + y &\leq 4 \\ 4x + 3y &\leq 12 \\ -x + y &\geq 1 \\ x + y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

8. Write the constraints associated with the solution space shown in Figure 22 and identify the redundant constraints.

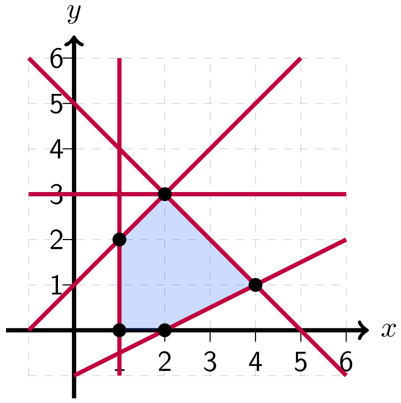


Figure 22

9. Consider the following problem:

$$\begin{array}{ll} \textbf{max} & z = 6x_1 - 2x_2 \\ \textbf{s.t.} & 3x_1 - x_2 \leq 6 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

Show graphically that at the optimal solution, the variables x_1 and x_2 can be increased indefinitely while the value of the objective function remains constant.

10. Consider the following problem:

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 2 \\ & 3x_1 + 4x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

Show graphically that the problem has no feasible optimal solution.

11. Solve the following problem by *inspection* without graphing the feasible region.

$$\begin{array}{ll} \textbf{min} & w = 5x_1 + 2x_2 \\ \textbf{s.t.} & x_1 + x_2 = 5 \\ & 7x_1 - 5x_2 = -1 \\ & x_1, x_2 \geq 0 \end{array}$$

12. Solve the following problems graphically.

(a) **min** $w = 4x_1 + x_2$
s.t. $3x_1 + x_2 \geq 6$
$4x_1 + x_2 \geq 12$
$x_1 \geq 2$
$x_1, x_2 \geq 0$

(b) **max** $z = 5x_1 - x_2$
s.t. $2x_1 + 3x_2 \geq 12$
$x_1 - 3x_2 \geq 0$
$x_1, x_2 \geq 0$

(c) **min** $w = 5x_1 + x_2$
s.t. $2x_1 + x_2 \geq 6$
$x_1 + x_2 \geq 4$
$x_1 + 5x_2 \geq 10$
$x_1, x_2 \geq 0$

(d) **max** $z = -4x_1 + x_2$
s.t. $3x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 12$
$x_1 \geq 0$ and x_2 urs

(e) **max** $z = x_1 + 5x_2$
s.t. $x_1 - 3x_2 \geq 0$
$x_1 + x_2 \leq 8$
$x_1, x_2 \geq 0$

(f) **min** $w = 4x_1 + x_2$
s.t. $3x_1 + x_2 \geq 10$
$x_1 + x_2 \geq 5$
$x_1 \geq 3$
$x_1, x_2 \geq 0$

2 The Simplex Method

Thus far we have used a geometric approach to solve certain LP problems. We have observed, in Chapter 1, that this procedure is limited to problems of two or three variables. The *simplex algorithm* is essentially algebraic in nature and is more efficient than its geometric counterpart.

2.1 The Idea of the Simplex Method

The simplex method in linear programming translates the geometric concept of corner points into an algebraic approach. It begins by converting all constraints into a standard form, where inequalities are expressed as equations. This results in a system with more variables than equations, leading to infinitely many solutions.

Extreme points (or corner points) of the solution space correspond to **basic solutions**, which are found by setting some variables to zero and solving for the rest. The simplex method starts with an initial basic solution and systematically moves to other basic solutions that improve the objective function's value. This process continues until the optimal solution is identified, at which point the algorithm terminates.

2.2 Converting an LP to Standard Form

We have seen that an LP can have both equality and inequality constraints. It also can have variables that are required to be nonnegative as well as those allowed to be unrestricted in sign (urs). The development of the simplex method computations is facilitated by imposing two requirements on the LP model:

1. All the constraints are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

An LP in this form is said to be in **standard form**. To convert an LP into standard form, we do the following steps.

1. If the right-hand of a constraint is negative, multiply both sides of the constraint by -1. This multiplication will convert a $\leq$ sign to $\geq$ and vise versa.
2. To convert a $\leq$ inequality to an equation, a nonnegative **slack** variable (unused amount) is added to the left-hand side of the constraint. For example, the constraint $3x_1 + 2x_2 \leq 12$ is converted into an equation as

$$3x_1 + 2x_2 + s = 12\ ; \quad s \geq 0.$$

3. Conversion from $\geq$ to $=$ is achieved by subtracting a nonnegative **surplus** (excess amount) variable from the left-hand side of the inequality. For

example, the surplus variable e converts the constraint $5x_1 + 3x_2 \geq 4$ to the equation

$$5x_1 + 3x_2 - e = 4 \; ; \quad e \geq 0.$$

4. An unrestricted variable x_i can be presented in terms of two nonnegative variables by using the substitution

$$x_i = y_{i1} - y_{i2} \; ; \quad y_{i1}, y_{i2} \geq 0$$

The substitution must be effected throughout all the constraints and the objective function. In the optimal LP solution only one of the two variables y_{i1} and y_{i2} can assume a positive value, but never both. Thus, when $y_{i1} > 0$, $y_{i2} = 0$ and vice versa. For example, if $x_i = 4$ then $y_{i1} = 4$ and $y_{i2} = 0$, and if $x_i = -4$ then $y_{i1} = 0$ and $y_{i2} = 4$.

Example 2.1. Write the following LP problem in standard form.

$$\begin{array}{ll} \textbf{min} & w = 2x_1 + 3x_2 \\ \textbf{s.t.} & x_1 + x_2 = 10 \\ & -2x_1 + 3x_2 \leq -5 \\ & 7x_1 - 4x_2 \leq 6 \\ & x_1 \text{ urs}, \; x_2 \geq 0 \end{array}$$

Solution: The following changes must be effected.

1. Multiply both sides of the second constraint by -1 to get $2x_1 - 3x_2 \geq 5$, then subtract excess variable $e_2 \geq 0$ from the left-hand side of the constraint.
2. Add a slack variable $s_3 \geq 0$ to the left-hand side of the third constraint.
3. Substitute $x_1 = y_{11} - y_{12}$, where $y_{11}, y_{12} \geq 0$, in the objective function and all the other constraints.

Thus we get the standard form as

$$\begin{array}{ll} \textbf{min} & w = 2y_{11} - 2y_{12} + 3x_2 \\ \textbf{s.t.} & y_{11} - y_{12} + x_2 = 10 \\ & 2y_{11} - 2y_{12} - 3x_2 - e_2 = 5 \\ & 7y_{11} - 7y_{12} - 4x_2 + s_3 = 6 \\ & y_{11}, y_{12}, x_2, e_2, s_3 \geq 0 \end{array}$$

Exercise 2.1.

1. Convert the following LP to the standard form.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 3x_2 + 5x_3 \\ \textbf{s.t.} & x_1 + x_2 - x_3 \geq -5 \\ & -6x_1 + 7x_2 - 9x_3 \leq 4 \\ & x_1 + x_2 + 4x_3 = 10 \\ & x_1, x_2 \geq 0, \; x_3 \text{ urs} \end{array}$$

2. Consider the inequality

$$22x_1 - 4x_2 \geq -7$$

Show that multiplying both sides of the inequality by -1 and then converting the resulting inequality into an equation is the same as converting it first to an equation and then multiplying both sides by -1.

3. The substitution $x = y_1 - y_2$ is used in an LP to replace unrestricted x by the two nonnegative variables y_1 and y_2. If x assumes the respective values -6, 10, and 0, determine the associated optimal values of y_1 and y_2 in each case.

2.3 Basic Feasible Solutions

Suppose we have converted an LP with m constraints into standard form. Assuming that the standard form contains n variables (labeled for convenience $x_1, x_2, \cdots, x_n$), where $n \geq m$, the standard form for such an LP is

$$\begin{array}{ll} \textbf{max (or min)} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \textbf{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_1, x_2, \cdots, x_n \geq 0 \end{array}$$

If we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

the constraints for the LP may be written as the system of equations $A\mathbf{x} = \mathbf{b}$.

Definition 2.1. A *basic solution* to $A\mathbf{x} = \mathbf{b}$ is obtained by setting $n-m$ variables (the nonbasic variables, or NBV) equal to 0 and solving for the values of the remaining m variables (the basic variables, or BV), provided the resulting solution is unique.

Note 7. The maximum number of corner points is $C_m^n = \dfrac{n!}{m!\,(n-m)!}$.

Of course, the different choices of nonbasic variables will lead to different basic solutions. To illustrate, we find all the basic solutions to the following system of two equations ($m = 2$) in three variables ($n = 3$):

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

We begin by choosing a set of $n - m = 3 - 2 = 1$ nonbasic variable, and note that there are $C_2^3 = 3$ choices of this nonbasic variable.

- If NBV= $\{x_3\}$, then BV= $\{x_1, x_2\}$. We obtain the values of the basic variables by setting $x_3 = 0$ and solving

  $$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 &= -1 \end{aligned}$$

  We find that $x_1 = 2$, $x_2 = 1$. Thus, $(x_1, x_2, x_3) = (2, 1, 0)$ is a basic solution to the system.
- If NBV= $\{x_2\}$, then BV= $\{x_1, x_3\}$. We obtain the values of the basic variables by setting $x_2 = 0$ and we find that $x_1 = 3$, $x_3 = -1$. Thus, $(x_1, x_2, x_3) = (3, 0, -1)$ is a basic solution to the system.
- If NBV= $\{x_1\}$, then BV= $\{x_2, x_3\}$. We obtain the values of the basic variables by setting $x_1 = 0$ and solving

  $$\begin{aligned} x_2 \phantom{{}+x_3} &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

  We find that $x_2 = 3$, $x_3 = 2$. Thus, $(x_1, x_2, x_3) = (0, 3, 2)$ is a basic solution to the system.

The following table provides all the basic and nonbasic solutions of the above linear system.

NBVs	BVs	Basic Solution
x_1	x_2, x_3	$3, 2$
x_2	x_1, x_3	$3, -1$
x_3	x_1, x_2	$2, 1$

Note 8. Some sets of m variables do not yield a basic solution. For example, consider the following linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + x_3 &= 3 \end{aligned}$$

If we choose NBV= $\{x_3\}$ and BV= $\{x_1, x_2\}$, the corresponding basic solution would be obtained by solving

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 3 \end{aligned}$$

Because this system has no solution, there is no basic solution corresponding to BV$= \{x_1, x_2\}$.

Definition 2.2. Any basic solution to the constraints $A\mathbf{x} = \mathbf{b}$ of an LP in which all variables are nonnegative is a *basic feasible solution* (or **bfs**).

For example, for an LP with the constraints given by

$$\begin{aligned} x_1 + x_2 \quad\quad &= 3 \\ - x_2 + x_3 &= -1 \end{aligned}$$

the basic solutions $x_1 = 2, x_2 = 1, x_3 = 0$, and $x_1 = 0, x_2 = 3, x_3 = 2$ are basic feasible solutions, but the basic solution $x_1 = 3, x_2 = 0, x_3 = -1$ fails to be a feasible solution (because $x_3 < 0$).

Theorem 2.1. *A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.*

Exercise 2.2.

1. Consider the following LP:

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 3x_2 \\ \textbf{s.t.} & x_1 + 3x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

   (a) Express the problem in equation form.
   (b) Determine all the basic solutions of the problem, and classify them as feasible and infeasible.
   (c) Use direct substitution in the objective function to determine the optimum basic feasible solution.
   (d) Verify graphically that the solution obtained in (c) is the optimum LP solution, hence, conclude that the optimum solution can be determined algebraically by considering the basic feasible solutions only.
   (e) Show how the *infeasible* basic solutions are represented on the graphical solution space.

2. Determine the optimum solution for each of the following LPs by enumerating all the basic solutions.

(a)
$$\begin{array}{ll}
\textbf{max} & z = 2x_1 - 4x_2 + 5x_3 - 6x_4 \\
\textbf{s.t.} & x_1 + 5x_2 - 2x_3 + 8x_4 \leq 2 \\
 & -x_1 + 2x_2 + 3x_3 + 4x_4 \leq 1 \\
 & x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

(b)
$$\begin{array}{ll}
\textbf{min} & w = x_1 + 2x_2 - 3x_3 - 2x_4 \\
\textbf{s.t.} & x_1 + 2x_2 - 3x_3 + x_4 = 4 \\
 & x_1 + 2x_2 + x_3 + 2x_4 = 4 \\
 & x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

3. Show algebraically that all the basic solutions of the following LP are infeasible.

$$\begin{array}{ll}
\textbf{max} & z = x_1 + x_2 \\
\textbf{s.t.} & x_1 + 2x_2 \leq 3 \\
 & 2x_1 + x_2 \geq 8 \\
 & x_1, x_2 \geq 0
\end{array}$$

4. Consider the following LP:

$$\begin{array}{ll}
\textbf{max} & z = x_1 + 3x_2 \\
\textbf{s.t.} & x_1 + x_2 \leq 2 \\
 & -2x_1 + x_2 \leq 4 \\
 & x_1 \text{ urs}, \ x_2 \geq 0
\end{array}$$

   (a) Determine all the basic feasible solutions of the problem.
   (b) Use direct substitution in the objective function to determine the best basic solution.
   (c) Solve the problem graphically, and verify that the solution obtained in (b) is the optimum.

2.4 The Simplex Algorithm

Rather than enumerating *all* the basic solutions (corner points) of the LP problem, as we did in section (1.2), the simplex method investigates only a "select few" of these solutions. This section describes the iterative nature of the method, and provides the computational details of the simplex algorithm.

Before describing the simplex algorithm in general terms, we need to define the concept of an adjacent basic feasible solution.

Definition 2.3. For any LP with m constraints, two basic feasible solutions are said to be *adjacent* if their sets of basic variables have $m - 1$ basic variables in common.

For example, in Figure 23, two basic feasible solutions will be adjacent if they have $2 - 1 = 1$ basic variable in common. Thus, the bfs corresponding to point B in Figure 23 is adjacent to the bfs corresponding to point C but is not adjacent to bfs D. Intuitively, two basic feasible solutions are adjacent if they both lie on the same edge of the boundary of the feasible region.

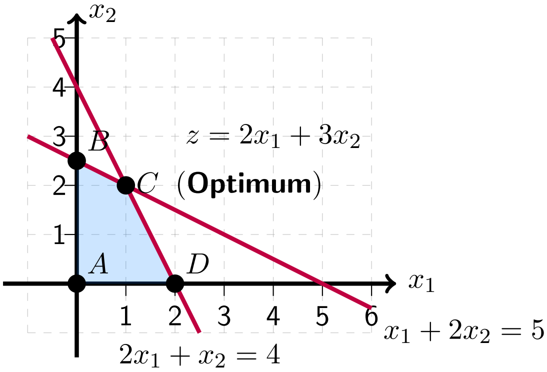


Figure 23

2.4.1 Iterative Nature of the Simplex Method

Figure 23 provides the solution space of an LP. For the sake of standardizing the algorithm, the simplex method always starts at the origin where all the decision variables, x_i are zero. In Figure 23, point A is the origin $(x_1 = x_2 = 0)$ and the associated objective value, z, is zero. The logical question now is whether an increase in the values of nonbasic x_1 and x_2 above their current zero values can improve (increase) the value of z. We can answer this question by investigating the objective function:

$$\textbf{max } z = 2x_1 + 3x_2$$

An increase in x_1 or x_2 (or both) above their current zero values will improve the value of z. The design of the simplex method does not allow simultaneous increases in variables. Instead, it targets the variables one at a time. The variable slated for increase is the one with the largest rate of improvement in z. In the present example, the rate of improvement in the value of z is 2 for x_1 and 3 for x_2. We thus elect to increase x_2 (the variable with the largest rate of improvement among all nonbasic variables). Figure 23 shows that the value of x_2 must be increased until corner point B is reached (recall that stopping short of corner point B is not an option because a candidate for the optimum must be a corner point). At point B, the simplex method, as will be explained later, will then increase the value of x_1 to reach the improved corner point C, which is the optimum.

The path of the simplex algorithm always connects adjacent corner points. In the present example the path to the optimum is $A \to B \to C$. Each corner point along the path is associated with an iteration. It is important to note that the simplex method always moves alongside the edges of the solution space, which means that the method does not cut across the solution space. For example, the simplex algorithm cannot go from A to C directly since they are not adjacent.

2.4.2 Computational Details of the Simplex Algorithm

We now describe how the simplex algorithm can be used to solve LPs in which the goal is to maximize the objective function. The solution of minimization problems is discussed in Section 2.5. The simplex algorithm proceeds as follows:

Step 1: Convert the LP to standard form. Then, write the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in the form

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0.$$

We call this format the *row 0 version* of the objective function (row 0 for short).

Step 2: Obtain a bfs (if possible) from the standard form. This is easy if all the constraints are $\leq$ with nonnegative right-hand sides. Then the slack variable s_i may be used as the basic variable for row i. If no bfs is readily apparent, then use the technique discussed in Section 2.6 to find a bfs.

Step 3: Determine whether the current bfs is optimal. If all nonbasic variables have nonnegative coefficients in row 0, then the current bfs is optimal. If any variables in row 0 have negative coefficients, then choose the variable with the most negative coefficient in row 0 to enter the basis. We call this variable the *entering* variable.

Step 4: If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value. When entering a variable into the basis, compute the ratio

$$\frac{\text{Right-hand side of constraint}}{\text{Coefficient of entering variable in constraint}}$$

for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called the *winner of the ratio test*. The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.

Step 5: Use elementary row operations (EROs) to find the new bfs with the better objective function value by making the entering variable a basic variable (has coefficient 1 in pivot row, and 0 in other rows) in the constraint that wins the ratio test. Go back to step 3.

2.4.3 Representing the Simplex Tableau

The tabular form of the simplex method records only the essential information:

- the coefficients of the variables,
- the constants on the right-hand sides of the equations,
- the basic variable appearing in each equation.

This saves writing the symbols for the variables in each of the equations, but what is even more important is the fact that it permits highlighting the numbers involved in arithmetic calculations and recording the computations compactly. For example, the form

$$\begin{aligned} z - 2x_1 - 3x_2 \qquad\qquad &= 0 \\ 2x_1 + \ \ x_2 + s_1 \qquad &= 4 \\ x_1 + 2x_2 \qquad + s_2 &= 5 \end{aligned}$$

would be written in abbreviated form as shown in the following table.

Basic	x_1	x_2	s_1	s_2	RHS
z	-2	-3	0	0	0
s_1	2	1	1	0	4
s_2	1	2	0	1	5

The layout of the simplex tableau automatically provides the solution at the starting iteration. The solution starts at the origin $(x_1, x_2) = (0, 0)$, thus defining (x_1, x_2) as the nonbasic variables and (s_1, s_2) as the basic variables. The associated objective z and the basic variables (s_1, s_2) are listed in the leftmost Basic-column. Their values, $z = 0$, $s_1 = 4$, $s_2 = 5$ appearing in the rightmost Solution-column, are given directly by the right-hand sides of the model's equations (a convenient consequence of starting at the origin). The result can be seen by setting the nonbasic variables (x_1, x_2) equal to zero in all the equations, and also by noting the special identity-matrix arrangement of the constraint coefficients of the basic variables (all diagonal elements are 1, and all off-diagonal elements are 0).

Example 2.2. Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 3x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: By adding slack variables s_1 and s_2, respectively, we obtain the LP in standard form:

$$\begin{aligned} \textbf{max} \quad & z - 2x_1 - 3x_2 = 0 \\ \textbf{s.t.} \quad & 2x_1 + x_2 + s_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

			↓				
Iteration [0]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	-2	-3	0	0	0	
	s_1	2	1	1	0	4	Ratio= $4/1 = 4$
←	s_2	1	2	0	1	5	Ratio= $5/2 = 2.5$

		↓					
Iteration [1]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	$-1/2$	0	0	$3/2$	$15/2$	
←	s_1	$3/2$	0	1	$-1/2$	$3/2$	Ratio= $3/2 \div 3/2 = 1$
	x_2	$1/2$	1	0	$1/2$	$5/2$	Ratio= $5/2 \div 1/2 = 5$

Iteration [2]	Basic	x_1	x_2	s_1	s_2	RHS
	z	0	0	$1/3$	$4/3$	8
	x_1	1	0	$2/3$	$-1/3$	1
	x_2	0	1	$-1/3$	$2/3$	2

Optimal Tableau

$z = 8$

$x_1 = 1, x_2 = 2$

$s_1 = 0, s_2 = 0$

Example 2.3. Solve the following LP problem using the simplex method.

$$\begin{aligned} \textbf{max} \quad & z = 4x_1 + 4x_2 \\ \textbf{s.t.} \quad & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution: By adding slack variables s_1, s_2, s_3 and s_4, respectively, we obtain the LP in standard form:

$$\begin{aligned} \textbf{max} \quad & z - 4x_1 - 4x_2 = 0 \\ \textbf{s.t.} \quad & 6x_1 + 4x_2 + s_1 = 24 \\ & x_1 + 2x_2 + s_2 = 6 \\ & -x_1 + x_2 + s_3 = 1 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below. Note that we can choose to enter either x_1 or x_2 into the basis. We arbitrarily choose to enter x_1 into basis.

Iteration [0]

		↓						
	Basic	x_1	x_2	s_1	s_2	s_3	RHS	
	z	-4	-4	0	0	0	0	
←	s_1	6	4	1	0	0	24	Ratio$= 24/6 = 4$
	s_2	1	2	0	1	0	6	Ratio$= 6/1 = 6$
	s_3	-1	1	0	0	1	1	✗

Iteration [1]

			↓					
	Basic	x_1	x_2	s_1	s_2	s_3	RHS	
	z	0	$-4/3$	$2/3$	0	0	16	
	x_1	1	$2/3$	$1/6$	0	0	4	Ratio$= 4 \div 2/3 = 6$
←	s_2	0	$4/3$	$-1/6$	1	0	2	Ratio$= 2 \div 4/3 = 3/2$
	s_3	0	$5/3$	$1/6$	0	1	5	Ratio$= 5 \div 5/3 = 3$

Iteration [2]

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	$1/2$	1	0	18
x_1	1	0	$1/4$	$-1/2$	0	3
x_2	0	1	$-1/8$	$3/4$	0	$3/2$
s_3	0	0	$3/8$	$-5/4$	1	$5/2$

Optimal Tableau

$z = 18$

$x_1 = 3, x_2 = 3/2$

$s_1 = 0, s_2 = 0$

$s_3 = 5/2, s_4 = 1/2$

Example 2.4. Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \textbf{max} & z = x_1 + 3x_2 \\ \textbf{s.t.} & x_1 + x_2 \leq 2 \\ & -x_1 + x_2 \leq 4 \\ & x_1 \geq 0,\ x_2 \text{ urs} \end{array}$$

Solution: By assuming $x_2 = y_1 - y_2$ and then adding slack variables s_1 and s_2, respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \textbf{max} & z - x_1 - 3y_1 + 3y_2 = 0 \\ \textbf{s.t.} & x_1 + y_1 - y_2 + s_1 = 2 \\ & -x_1 + y_1 - y_2 + s_2 = 4 \\ & x_1, y_1, y_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

Iteration [0]

			↓					
	Basic	x_1	y_1	y_2	s_1	s_2	RHS	
	z	-1	-3	3	0	0	0	
←	s_1	1	1	-1	1	0	2	Ratio$= 2/1 = 2$
	s_2	-1	1	-1	0	1	4	Ratio$= 4/1 = 4$

Iteration [1]

Basic	x_1	y_1	y_2	s_1	s_2	RHS
z	2	0	0	3	0	6
y_1	1	1	-1	1	0	2
s_2	-2	0	0	-1	1	2

Optimal Tableau

$z = 6$

$x_1 = 0,\ x_2 = 2$

$s_1 = 0,\ s_2 = 2$

Note that from the optimal tableau we have $y_1 = 2$ and $y_2 = 0$, so that $x_2 = y_1 - y_2 = 2 - 0 = 2$.

Exercise 2.3.

1. Use the simplex algorithm to solve the following problems.

(a)
$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 3x_2 \\ \textbf{s.t.} & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

(b)
$$\begin{array}{ll} \textbf{max} & z = 2x_1 - x_2 + x_3 \\ \textbf{s.t.} & 3x_1 + x_2 + x_3 \leq 60 \\ & x_1 - x_2 + 2x_3 \leq 10 \\ & x_1 + x_2 - x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(c)
$$\begin{array}{ll} \textbf{max} & z = x_1 - x_2 \\ \textbf{s.t.} & 4x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

(d)
$$\begin{array}{ll} \textbf{max} & z = x_1 + x_2 + x_3 \\ \textbf{s.t.} & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + 2x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

2. Solve the following problem by *inspection*, and justify the method of solution in terms of the basic solutions of the simplex method.

$$\begin{array}{ll} \textbf{max} & z = 5x_1 - 6x_2 + 3x_3 - 5x_4 + 12x_5 \\ \textbf{s.t.} & x_1 + 3x_2 + 5x_3 + 6x_4 + 3x_5 \leq 30 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

2.5 Solving Minimization Problem

There are two different ways that the simplex algorithm can be used to solve minimization problems.

Method (1) Multiply the objective function for the min problem by -1 and solve the problem as a maximization problem with objective function $(-w)$. The optimal solution to the max problem will give you the optimal solution to the min problem where

$$\begin{bmatrix} \text{optimal objective function} \\ \text{value for } \textit{min} \text{ problem} \end{bmatrix} = - \begin{bmatrix} \text{optimal objective function} \\ \text{value for } \textit{max} \text{ problem} \end{bmatrix}$$

Example 2.5. Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \textbf{min} & w = 2x_1 - 3x_2 \\ \textbf{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: The optimal solution to the LP is the point (x_1, x_2) in the feasible region for the LP that makes $w = 2x_1 - 3x_2$ the smallest. Equivalently, we may say that the optimal solution to the LP is the point in the feasible region that makes $z = -w = -2x_1 + 3x_2$ the largest. This means that we can find the optimal solution to the LP by solving:

$$\begin{array}{ll} \textbf{max} & z = -2x_1 + 3x_2 \\ \textbf{s.t.} & x_1 + x_2 \le 4 \\ & x_1 - x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{array}$$

By adding slack variables s_1 and s_2, respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \textbf{max} & z + 2x_1 - 3x_2 = 0 \\ \textbf{s.t.} & x_1 + x_2 + s_1 = 4 \\ & x_1 - x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

			↓				
Iteration [0]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	2	-3	0	0	0	
←	s_1	1	1	1	0	4	Ratio$= 4/1 = 4$
	s_2	1	-1	0	1	6	✗

Iteration [1]	Basic	x_1	x_2	s_1	s_2	RHS	**Optimal Tableau**
	z	5	0	3	0	12	$w = -z = -12$
	x_2	1	1	1	0	4	$x_1 = 0,\ x_2 = 4$
	s_2	2	0	1	1	10	$s_1 = 0,\ s_2 = 10$

Method (2) A simple modification of the simplex algorithm can be used to solve min problems directly. Modify *Step 3* of the simplex as follows: If all nonbasic variables in row 0 have nonpositive coefficients, then the current bfs is optimal. If any nonbasic variable in row 0 has a positive coefficient, choose the variable with the "most positive" coefficient in row 0 to enter the basis. This modification of the simplex algorithm works because increasing a nonbasic variable with a positive coefficient in row 0 will decrease w. If we use this method to solve the LP in example (2.5), then after adding slack variables s_1 and s_2, respectively, we obtain the LP in standard form:

$$\begin{array}{ll} \textbf{min} & w - 2x_1 + 3x_2 = 0 \\ \textbf{s.t.} & x_1 + x_2 + s_1 = 4 \\ & x_1 - x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below. Note that, because x_2 has the most positive coefficient in row 0, we enter x_2 into the basis.

			↓				
Iteration [0]	Basic	x_1	x_2	s_1	s_2	RHS	
	w	-2	3	0	0	0	
←	s_1	1	1	1	0	4	Ratio= $4/1 = 4$
	s_2	1	-1	0	1	6	✗

Iteration [1]	Basic	x_1	x_2	s_1	s_2	RHS	**Optimal Tableau**
	w	-5	0	-3	0	-12	$w = -12$
	x_2	1	1	1	0	4	$x_1 = 0,\ x_2 = 4$
	s_2	2	0	1	1	10	$s_1 = 0,\ s_2 = 10$

Exercise 2.4. Use the simplex algorithm to solve the following problems.

1. **min** $w = 4x_1 - x_2$
   s.t. $2x_1 + x_2 \leq 8$
   $x_2 \leq 5$
   $x_1 - x_2 \leq 4$
   $x_1, x_2 \geq 0$

2. **min** $w = x_1 - x_2$
   s.t. $x_1 - x_2 \leq 1$
   $x_1 + x_2 \leq 2$
   $x_1, x_2 \geq 0$

3. **min** $w = 2x_1 - 5x_2$
   s.t. $3x_1 + 8x_2 \leq 12$
   $2x_1 + 3x_2 \leq 6$
   $x_1, x_2 \geq 0$

4. **min** $w = -3x_1 + 8x_2$
   s.t. $4x_1 + 2x_2 \leq 12$
   $2x_1 + 3x_2 \leq 6$
   $x_1, x_2 \geq 0$

5. **min** $w = 5x_1 + 4x_2 + 6x_3 - 8x_4$
   s.t. $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40$
   $2x_1 - 2x_2 + x_3 + 2x_4 \leq 8$
   $4x_1 - 2x_2 + x_3 - x_4 \leq 10$
   $x_1, x_2, x_3, x_4 \geq 0$

2.6 Artificial Starting Solution and the Big $M-$Method

Recall that the simplex algorithm requires a starting bfs. In all the problems we have solved so far, we found a starting bfs by using the slack variables as our basic variables. If an LP has any $(\geq)$ or $(=)$ constraints, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the *Big $M-$method* may be used to solve the problem. The Big $M-$method is a version of the simplex algorithm that first finds a bfs by adding "*artificial*" variables to the problem. The objective function of the original LP must be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

The big $M-$method starts with the LP in equation form. If equation i does not have a slack (or a variable that can play the role of a slack), an artificial variable, a_i, is added to form a starting solution similar to the all-slack basic solution. However, because the artificial variables are not part of the original problem, a modeling "trick" is needed to force them to zero value by the time the optimum iteration is reached (assuming the problem has a feasible solution). The desired goal is achieved by assigning a penalty defined as:

$$\begin{array}{c}\text{Artificial variable objective}\\ \text{function coefficient}\end{array} = \begin{cases} -M & \text{in max problems} \\ M & \text{in min problems}\end{cases}$$

where M is a sufficiently large positive value (mathematically, $M \to \infty$).

Example 2.6. Solve the following LP problem using the simplex method.

$$\begin{array}{ll}\textbf{min} & w = 4x_1 + x_2 \\ \textbf{s.t.} & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0\end{array}$$

Solution: To convert the constraint to equations, use e_2 as a surplus in the second constraint and s_3 as a slack in the third constraint.

$$\begin{aligned} 3x_1 + x_2 \phantom{{}- e_2 + s_3} &= 3 \\ 4x_1 + 3x_2 - e_2 \phantom{{}+ s_3} &= 6 \\ x_1 + 2x_2 \phantom{{}- e_2} + s_3 &= 4 \end{aligned}$$

The third equation has its slack variable, s_3, but the first and second equations do not. Thus, we add the artificial variables a_1 and a_2 in the first two equations and penalize them in the objective function with $Ma_1 + Ma_2$ (because we are minimizing). The resulting LP becomes

$$\begin{array}{ll}\textbf{min} & w = 4x_1 + x_2 + Ma_1 + Ma_2 \\ \textbf{s.t.} & 3x_1 + x_2 + a_1 = 3 \\ & 4x_1 + 3x_2 - e_2 + a_2 = 6 \\ & x_1 + 2x_2 + s_3 = 4 \\ & x_1, x_2, s_3, e_2, a_1, a_2 \geq 0\end{array}$$

After writing the objective function as $w - 4x_1 - x_2 - Ma_1 - Ma_2 = 0$, the initial tableau will be

Iteration [0]

Basic	x_1	x_2	s_3	e_2	a_1	a_2	RHS
w	-4	-1	0	0	$-M$	$-M$	0
a_1	3	1	0	0	1	0	3
a_2	4	3	0	-1	0	1	6
s_3	1	2	1	0	0	0	4

Before proceeding with the simplex method computations, row 0 must be made consistent with the rest of the tableau. The right−hand side of row 0 in the tableau currently shows $w = 0$. However, given the nonbasic solution $x1 = x2 = e_2 = 0$, the current basic solution is $a_1 = 3$, $a_2 = 6$, and $s_3 = 4$ yields

$$w = (4 \times 0) + (1 \times 0) + (3 \times M) + (6 \times M) = 9M \neq 0.$$

The inconsistency stems from the fact that a_1 and a_2 have nonzero coefficients in row 0. To eliminate the inconsistency, we use EROs. The modified tableau thus becomes (verify!):

Iteration [0]

Basic	x_1	x_2	s_3	e_2	a_1	a_2	RHS
w	$7M-4$	$4M-1$	0	$-M$	0	0	$9M$
a_1	3	1	0	0	1	0	3
a_2	4	3	0	-1	0	1	6
s_3	1	2	1	0	0	0	4

The last tableau is ready for the application of the simplex optimality and the feasibility conditions. Because the objective function is minimized, the variable x_1 having the most positive coefficient in the row 0 enters the solution. The minimum ratio of the feasibility condition specifies a_1 as the leaving variable. All tableaus until the optimal solution is reached are shown below.

Iteration [0]

	Basic	↓ x_1	x_2	s_3	e_2	a_1	a_2	RHS	
	w	$7M-4$	$4M-1$	0	$-M$	0	0	$9M$	
←	a_1	3	1	0	0	1	0	3	Ratio= $3/3 = 1$
	a_2	4	3	0	-1	0	1	6	Ratio= $6/4 = 3/2$
	s_3	1	2	1	0	0	0	4	Ratio= $4/2 = 2$

Iteration [1]

	Basic	x_1	↓ x_2	s_3	e_2	a_1	a_2	RHS	
	w	0	$(1+5M)/3$	0	$-M$	$(4-7M)/3$	0	$4+2M$	
	x_1	1	$1/3$	0	0	$1/3$	0	1	Ratio= $1 \div 1/3 = 3$
←	a_2	0	$5/3$	0	-1	$-4/3$	1	2	Ratio= $2 \div 5/3 = 6/5$
	s_3	0	$5/3$	1	0	$-1/3$	0	3	Ratio= $3 \div 5/3 = 9/5$

Iteration [2]

	Basic	x_1	x_2	s_3	↓ e_2	a_1	a_2	RHS	
	w	0	0	0	$1/5$	$8/5 - M$	$-1/5 - M$	$18/5$	
	x_1	1	0	0	$1/5$	$3/5$	$-1/5$	$3/5$	Ratio= $3/5 \div 1/5 = 3$
	x_2	0	1	0	$-3/5$	$-4/5$	$3/5$	$6/5$	✗
←	s_3	0	0	1	1	1	-1	1	Ratio= $1/1 = 1$

Iteration [3]

Basic	x_1	x_2	s_3	e_2	a_1	a_2	RHS
w	0	0	$-1/5$	0	$7/5 - M$	$-M$	$17/5$
x_1	1	0	$-1/5$	0	$2/5$	0	$2/5$
x_2	0	1	$3/5$	0	$-1/5$	0	$9/5$
e_2	0	0	1	1	1	-1	1

Optimal Tableau

$w = 17/5$

$x_1 = 2/5$, $x_2 = 9/5$

$e_2 = 1$

$a_1 = 0$, $a_2 = 0$

Note 9. From a computational standpoint, solving the problem on the computer requires replacing M with a *sufficiently large* numeric value. The result is an unnecessary layer of computational difficulty that can be avoided by substituting an appropriate numeric value for M (which is what we would do anyway if we use the computer). We break away from the long tradition of manipulating M algebraically and use a numerical substitution instead. The intent is to simplify the presentation without losing substance. What value of M should we use? The answer depends on the data of the original LP. Recall that the penalty M must be sufficiently large relative to the original objective coefficients to force the artificial variables to be zero (which happens only if a feasible solution exists). At the same time, since computers are the main tool for solving LPs, M should not be unnecessarily too large, as this may lead to serious round-off error. In the present example, the objective coefficients of x_1 and x_2 are 2 and 1, respectively, and it appears reasonable to set $M = 100$.

Example 2.7. Solve the following LP problem using the simplex method.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + x_2 \\ \textbf{s.t.} & x_1 + x_2 \leq 10 \\ & -x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: To convert the constraint to equations, use s_1 as a slack in the first constraint and e_2 as a surplus in the second constraint.

$$\begin{aligned} x_1 + x_2 + s_1 \qquad &= 10 \\ -x_1 + x_2 \qquad - e_2 &= \ 2 \end{aligned}$$

We add the artificial variables a_2 in the second equation and penalize it in the objective function with $-Ma_2 = -100a_2$ (because we are maximizing). The resulting LP becomes

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + x_2 - 100a_2 \\ \textbf{s.t.} & x_1 + x_2 + s_1 = 10 \\ & -x_1 + x_2 - e_2 + a_2 = 2 \\ & x_1, x_2, s_1, e_2, a_2 \geq 0 \end{array}$$

After writing the objective function as $z - 2x_1 - x_2 + 100a_2 = 0$, the initial tableau will be

Iteration [0]

Basic	x_1	x_2	s_1	e_2	a_2	RHS
z	-2	-1	0	0	100	0
s_1	1	1	1	0	0	10
a_2	-1	1	0	-1	1	2

Before proceeding with the simplex method computations, row 0 must be made consistent with the rest of the tableau. The inconsistency stems from the fact that a_2 has nonzero coefficients in row 0. To eliminate the inconsistency, we use EROs. The modified tableau and all other tableaus until the optimal solution is reached are:

Iteration [0]

	Basic	x_1	$\downarrow$ x_2	s_1	e_2	a_2	RHS	
	z	98	-101	0	100	0	-200	
	s_1	1	1	1	0	0	10	Ratio= $10/1 = 10$
$\leftarrow$	a_2	-1	1	0	-1	1	2	Ratio= $2/1 = 2$

Iteration [1]

	Basic	$\downarrow$ x_1	x_2	s_1	e_2	a_2	RHS	
	z	-3	0	0	-1	101	2	
$\leftarrow$	s_1	2	0	1	1	-1	8	Ratio= $8/2 = 4$
	x_2	-1	1	0	-1	1	2	✗

Iteration [2]

Basic	x_1	x_2	s_1	e_2	a_2	RHS
z	0	0	$3/2$	$1/2$	$199/2$	14
x_1	1	0	$1/2$	$1/2$	$-1/2$	4
x_2	0	1	$1/2$	$-1/2$	$1/2$	6

Optimal Tableau

$z = 14$

$x_1 = 4, x_2 = 6$

$s_1 = e_2 = a_2 = 0$

Example 2.8. Consider the problem.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 4x_2 + 4x_3 - 3x_4 \\ \textbf{s.t.} & x_1 + x_2 + x_3 \qquad = 4 \\ & x_1 + 4x_2 \qquad + x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

The variables x_3 and x_4 play the role of slack variables. So, without using any artificial variables, solve the problem with x_3 and x_4 as the starting basic variables.

Solution: The main difference here from the usual simplex is that x_3 and x_4 have nonzero objective coefficients in row 0: $z - 2x_1 - 4x_2 - 4x_3 + 3x_4 = 0$. To eliminate their coefficients, we use EROs. The initial tableaus and all following tableaus until the optimal solution is reached are shown below.

Iteration [0]

Basic	x_1	x_2	x_3	x_4	RHS
z	-2	-4	-4	3	0
x_3	1	1	1	0	4
x_4	1	4	0	1	8

Iteration [0]

	Basic	x_1	$\downarrow$ x_2	x_3	x_4	RHS	
	z	-1	-12	0	0	-8	
	x_3	1	1	1	0	4	Ratio= $4/1 = 4$
$\leftarrow$	x_4	1	4	0	1	8	Ratio= $8/4 = 2$

Iteration [1]

Basic	x_1	x_2	x_3	x_4	RHS
z	2	0	0	3	16
x_3	$3/4$	0	1	$-1/4$	2
x_2	$1/4$	1	0	$1/4$	2

Optimal Tableau

$z = 16$

$x_1 = 0,\ x_2 = 2$

$x_3 = 2,\ x_4 = 0$

Exercise 2.5.

1. Use the Big M-method to solve the following LPs:

(a)
$$\begin{array}{ll} \textbf{min} & w = 4x_1 + 4x_2 + x_3 \\ \textbf{s.t.} & x_1 + x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + x_2 + 3x_3 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(b)
$$\begin{array}{ll} \textbf{min} & w = x_1 + x_2 \\ \textbf{s.t.} & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + 2x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(c)
$$\begin{array}{ll} \textbf{min} & w = 2x_1 + 3x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 - x_2 \geq -1 \\ & x_1, x_2 \geq 0 \end{array}$$

(d)
$$\begin{array}{ll} \textbf{max} & z = 3x_1 + x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{array}$$

2. Solve the following problem using x_3 and x_4 as starting basic feasible variables. As in example (2.8), do not use any artificial variables.

$$\begin{array}{ll} \textbf{min} & z = 3x_1 + 2x_2 + 3x_3 + 2x_4 \\ \textbf{s.t.} & x_1 + 4x_2 + x_3 \qquad \geq 14 \\ & 2x_1 + x_2 \qquad + x_4 \geq 20 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

3. Consider the problem

$$\begin{array}{ll} \textbf{max} & z = x_1 + 5x_2 + 3x_3 \\ \textbf{s.t.} & x_1 + 2x_2 + x_3 = 6 \\ & 2x_1 - x_2 \qquad = 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The variable x_3 plays the role of a slack. Thus, no artificial variable is needed in the first constraint. In the second constraint, an artificial variable, a_2, is needed. Solve the problem using x_3 and a_2 as the starting variables.

2.7 Special Cases in the Simplex Method

This section considers four special cases that arise in the use of the simplex method.

2.7.1 Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one basic variable will be zero in the next iteration, and the new solution is said to be degenerate. This situation may reveal that the model has at least one redundant constraint.

Definition 2.4. An LP is *degenerate* if it has at least one bfs in which a basic variable is equal to zero.

If one of these degenerate basic variables retains its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero, so the value of the objective function must remain unchanged. However, if the objective function may remain the same rather than change at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually changing the objective function toward an optimal solution. This occurrence is called *cycling*.

Example 2.9. Solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 9x_2 \\ \textbf{s.t.} & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: By adding slack variables s_1 and s_2, we obtain the LP in standard form

$$\begin{array}{ll} \textbf{max} & z - 3x_1 - 9x_2 = 0 \\ \textbf{s.t.} & x_1 + 4x_2 + s_1 = 8 \\ & x_1 + 2x_2 + s_2 = 4 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

			↓				
Iteration [0]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	-3	-9	0	0	0	
←	s_1	1	4	1	0	8	Ratio$= 8/4 = 2$
	s_2	1	2	0	1	4	Ratio$= 4/2 = 2$

In iteration 0, s_1 and s_2 tie for the leaving variable, leading to degeneracy in iteration 1 because the basic variable s_2 assumes a zero value.

		↓					
Iteration [1]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	$-3/4$	0	$9/4$	0	18	
	x_2	$1/4$	1	$1/4$	0	2	Ratio$= 2 \div 1/4 = 8$
←	s_2	$1/2$	0	$-1/2$	1	0	Ratio$= 0 \div 1/2 = 0$

Iteration [2]	Basic	x_1	x_2	s_1	s_2	RHS
	z	0	0	$3/2$	$3/2$	18
	x_2	0	1	$1/2$	$-1/2$	2
	x_1	1	0	-1	2	0

Optimal Tableau

$z = 18$

$x_1 = 0,\ x_2 = 2$

$s_1 = 0,\ s_2 = 0$

The following example illustrates the occurrence of cycling in the simplex iterations and the possibility that the algorithm may never converge to the optimum solution.

Example 2.10. This example was authored by E.M. Beale[1]. Consider the following LP:

$$\begin{array}{ll}
\textbf{max} & C = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4 \\
\textbf{s.t.} & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \leq 0 \\
& \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \leq 0 \\
& x_3 \leq 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

Actually, the optimal solution of this example is $C = \frac{1}{20}$ when $x_1 = \frac{1}{25}$, $x_3 = 1$, and $x_2 = x_4 = 0$.

Note 10. There are several ways to solve the LP problem in example (2.10). We review these methods as follows.

1. **Computer Systems:** like `Excel Solver`, `LINDO` and `Mathematica`.
2. **Convert all the coefficients in the constraints to integer values by using proper multiples:**[2] this can be done by multiplying the first constraint in the original LP by $\mathrm{lcm}(4, 25) = 100$ and the second constrain by $\mathrm{lcm}(2, 50) = 50$. Then we write the LP in standard form.

$$\begin{array}{ll}
\textbf{max} & C - \frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 = 0 \\
\textbf{s.t.} & 25x_1 - 6000x_2 - 4x_3 + 900x_4 + s_1 = 0 \\
& 25x_1 - 4500x_2 - x_3 + 150x_4 + s_2 = 0 \\
& x_3 + s_3 = 1 \\
& x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0
\end{array}$$

3. **Bland's Rule for selecting entering and leaving variables[3].**
   (a) For the entering basic variable: Of all negative coefficients in the objective row (Row 0), choose the one with smallest subscript.
   (b) For the departing basic variable: When there is a tie between one or more ratios computed, choose the candidate for departing basic variable that has the smallest subscript.

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[1] Saul I. Gass, Sasirekha Vinjamuri. **Cycling in linear programming problems**. Computers & Operations Research 31 (2004)

[2] Hamdy A. Taha, **Operations Research: An Introduction, 9th Edition**, Prentice Hall. 2011. Call number in PU library: 658.4034 TAH.

[3] James Calvert and William Voxman, **Linear Programming, 1st Edition**, Harcourt Brace Jovanovich Publishers, 1989.

When we use Bland's rule to solve the LP in example (2.10), we name the slake variables s_1, s_2, s_3 as x_5, x_6, x_7 respectively.

4. **Lexicographic Rule for selecting an exiting variable**[4].
Given a basic feasible solution with basis $\mathbf{B}$, suppose that the nonbasic variable x_k is chosen to enter the basis (the most negative value in Row 0 for maximization LP). The index r of the variable x_B leaving the basis is determined as follows. Let

$$I_0 = \left\{ r : \frac{b_r}{y_{rk}} = \min_{0 \le i \le m} \left\{ \frac{b_i}{y_{ik}} : y_{ik} > 0 \right\} \right\}$$

If I_0 is a singleton, namely $I_0 = \{r\}$, then x_{B_r} leaves the basis. Otherwise, form I_1 as follows:

$$I_1 = \left\{ r : \frac{y_{r1}}{y_{rk}} = \min_{i \in I_0} \left\{ \frac{y_{i1}}{y_{ik}} \right\} \right\}$$

where y_{*1} is the first column of the $m \times m$ identity matrix. If I_1 is singleton, namely, $I_1 = \{r\}$, then x_{B_r} leaves the basis. Otherwise, form I_2, where, in general, I_j is formed from I_{j-1} as follows:

$$I_j = \left\{ r : \frac{y_{rj}}{y_{rk}} = \min_{i \in I_{j-1}} \left\{ \frac{y_{ij}}{y_{ik}} \right\} \right\}$$

where y_{*j} is the jth column of the $m \times m$ identity matrix. Eventually, for some $j \le m$, I_j will be a singleton. If $I_j = \{r\}$, then x_{B_r} leaves the basis.

2.7.2 Alternative Optima

Recall from example (1.13) of Section 1.3 that for some LPs, more than one extreme point is optimal. If an LP has more than one optimal solution, then we say that it has multiple or alternative optimal solutions. An LP problem may have an infinite number of alternative optima when the objective function is parallel to a nonredundant binding constraint. The existence of alternative can be detected in the optimal tableau by examining row 0 coefficients of the nonbasic variables. The zero coefficient of nonbasic x_j indicates that x_j can be made basic, altering the values of the basic variables without changing the value of z.

In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. If the example represents a product-mix situation, it may be advantageous to market two products instead of one.

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[4] Mokhtar S. Bazaraa, John J. Jarvis, Hanif D. Sher, **Linear Programming and Network Flows, 4th Edition**, John Wiley & Sons, Inc. 2010. Call number in PU library: 519.72 BAZ

Example 2.11. Solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + 4x_2 \\ \textbf{s.t.} & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: By adding slack variables s_1 and s_2, we obtain the LP in standard form

$$\begin{array}{ll} \textbf{max} & z - 2x_1 - 4x_2 = 0 \\ \textbf{s.t.} & x_1 + 2x_2 + s_1 = 5 \\ & x_1 + x_2 + s_2 = 4 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

			↓				
Iteration [0]	Basic	x_1	x_2	s_1	s_2	RHS	
	z	-2	-4	0	0	0	
←	s_1	1	2	1	0	5	Ratio= $5/2$
	s_2	1	1	0	1	4	Ratio= $4/1 = 4$

		↓						
Iteration [1]	Basic	x_1	x_2	s_1	s_2	RHS	**Optimal**	
	z	$\boxed{0}$	0	2	0	10	**Tableau**	
	x_2	$1/2$	1	$1/2$	0	$5/2$		Ratio= $5/2 \div 1/2 = 5$
←	s_2	$1/2$	0	$-1/2$	1	$3/2$		Ratio= $3/2 \div 3/2 = 3$

Iteration [2]	Basic	x_1	x_2	s_1	s_2	RHS	**Alternative**
	z	0	0	2	0	10	**Optima**
	x_2	0	1	1	-1	1	
	x_1	1	0	-1	2	3	

Mathematically, we can determine all the points (x_1, x_2) on the line segment joining the optimal solutions $\left(0, \frac{5}{2}\right)$ and $(3, 1)$ as follows:

$$\left.\begin{array}{l} x_1 = t(0) + (1-t)(3) = 3 - 3t \\ x_2 = t\left(\frac{5}{2}\right) + (1-t)(1) = 1 + \frac{3t}{2} \end{array}\right\} \; ; \; 0 \leq t \leq 1$$

2.7.3 Unbounded Solutions

In some LP models, as in example (1.15) of Section 1.3, the solution space is unbounded in at least one variable, meaning that variables may be increased indefinitely without violating any of the constraints. The associated objective value may also be unbounded in this case. An unbounded LP for a max problem occurs when a variable with a negative coefficient (positive for min LP) in row 0 has a nonpositive coefficient in each constraint.

An unbounded solution space may signal that the model is poorly constructed. The most likely irregularity in such models is that some key constraints have not been accounted for. Another possibility is that estimates of the constraint coefficients may not be accurate.

Example 2.12. Solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 2x_1 + x_2 \\ \textbf{s.t.} & x_1 - x_2 \leq 10 \\ & 2x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: By adding slack variables s_1 and s_2, we obtain the LP in standard form

$$\begin{array}{ll} \textbf{max} & z - 2x_1 - x_2 = 0 \\ \textbf{s.t.} & x_1 - x_2 + s_1 = 10 \\ & 2x_1 + s_2 = 40 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

The initial tableau and all following tableaus until the optimal solution is reached are shown below.

Iteration [0]

Basic	x_1	x_2	s_1	s_2	RHS
z	-2	-1	0	0	0
s_1	1	-1	1	0	10
s_2	2	0	0	1	40

In the starting tableau, both x_1 and x_2 have negative $z-$equation coefficients, meaning that an increase in their values will increase the objective value. Although x_1 should be the entering variable (it has the most negative $z-$coefficient), we note that all the constraint coefficients under x_2 are ≤ 0, meaning that x_2 can be increased indefinitely without violating any of the constraints. The result is that z can be increased indefinitely.

2.7.4 Nonexisting (or Infeasible) Solutions

LP models with inconsistent constraints have no feasible solution, see example (1.14) of Section 1.3. This situation does not occur if all the constraints are of the type $\leq$ with nonnegative right-hand sides because the slacks provide an obvious feasible solution. For other types of constraints, penalized artificial variables are used to start the solution. If at least one artificial variable is positive in the optimum iteration, then the LP has no feasible solution. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example 2.13. Solve the following LP problem.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 2 \\ & 3x_1 + 4x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: To convert the constraint to equations, use s_1 as a slack in the first constraint and e_2 as a surplus in the second constraint.

$$\begin{aligned} 2x_1 + \ \ x_2 + s_1 \qquad &= \ 2 \\ 3x_1 + 4x_2 \qquad - e_2 &= 12 \end{aligned}$$

We add the artificial variables a_2 in the second equation and penalize it in the objective function with $-Ma_2 = -100a_2$ (because we are maximizing). The resulting LP becomes

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 - 100a_2 \\ \textbf{s.t.} & 2x_1 + x_2 + s_1 = 2 \\ & 3x_1 + 4x_2 - e_2 + a_2 = 12 \\ & x_1, x_2, s_1, e_2, a_2 \geq 0 \end{array}$$

After writing the objective function as $z - 3x_1 - 2x_2 + 100a_2 = 0$, the initial tableau will be and all following tableaus until the optimal solution is reached are shown below.

Iteration [0]

Basic	x_1	x_2	s_1	e_2	a_2	RHS
z	-3	-2	0	0	$\boxed{100}$	0
s_1	2	1	1	0	0	2
a_2	3	4	0	-1	1	12

↓

Iteration [0]

	Basic	x_1	x_2	s_1	e_2	a_2	RHS	
	z	-303	-402	0	100	0	-1200	
←	s_1	2	1	1	0	0	2	Ratio$= 2/1 = 2$
	a_2	3	4	0	-1	1	12	Ratio$= 12/4 = 3$

Iteration [1]

Basic	x_1	x_2	s_1	e_2	a_2	RHS
z	501	0	402	100	0	-396
x_2	2	1	1	0	0	2
$\boxed{a_2}$	-5	0	-4	-1	1	4

Optimal Tableau

Optimum iteration 1 shows that the artificial variable a_2 is positive $(= 4)$, meaning that the LP is infeasible. The result is what we may call a *pseudo-optimal* solution.

Exercise 2.6.

1. Consider the following LP:

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 \\ \textbf{s.t.} & 4x_1 - x_2 \leq 4 \\ & 4x_1 + 3x_2 \leq 6 \\ & 4x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

   (a) Show that the associated simplex iterations are temporarily degenerate. How many iterations are needed to reach the optimum?

   (b) Verify the result by solving the problem graphically.

   (c) Interchange constraints (1) and (3) and resolve the problem. How many iterations are needed to solve the problem?

2. For the following LP, identify three alternative optimal basic solutions.

$$\begin{array}{ll} \textbf{max} & z = x_1 + 2x_2 + 3x_3 \\ \textbf{s.t.} & x_1 + 2x_2 + 3x_3 \leq 10 \\ & x_1 + x_2 \leq 5 \\ & x_1 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

3. Solve the following LP:

$$\begin{array}{ll} \textbf{max} & z = 2x_1 - x_2 + 3x_3 \\ \textbf{s.t.} & x_1 - x_2 + 5x_3 \leq 5 \\ & 2x_1 - x_2 + 3x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   From the optimal tableau, show that all the alternative optima are not corner points (i.e., nonbasic).

4. For the following LP, show that the optimal solution is degenerate and that none of the alternative solutions are corner points.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + x_2 \\ \textbf{s.t.} & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 - x_3 \leq 2 \\ & 7x_1 + 3x_2 - 5x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

5. Consider the LP:

$$\begin{array}{ll} \textbf{max} & z = 20x_1 + 5x_2 + x_3 \\ \textbf{s.t.} & 3x_1 + 5x_2 - 5x_3 \leq 50 \\ & x_1 \leq 10 \\ & x_1 + 3x_2 - 4x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   (a) By inspecting the constraints, determine the direction (x_1, x_2, x_3) in which the solution space is unbounded.

   (b) Without further computations, what can you conclude regarding the optimum objective value?

6. Consider the LP model

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + 2x_2 + 3x_3 \\ \textbf{s.t.} & 2x_1 + x_2 + x_3 \leq 4 \\ & 3x_1 + 4x_2 + 2x_3 \geq 16 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   Use hand computations to show that the optimal solution can include an artificial basic variable at zero level. Does the problem have a feasible optimal solution?

7. The following tableau represents a specific simplex iteration. All variables are nonnegative. The tableau is not optimal for either maximization or minimization. Thus, when a nonbasic variable enters the solution, it can either increase or decrease z or leave it unchanged, depending on the parameters of the entering nonbasic variable.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	0	-5	0	4	-1	-10	0	0	620
x_8	0	3	0	-2	-3	-1	5	1	12
x_3	0	1	1	3	1	0	3	0	6
x_1	1	-1	0	0	6	-4	0	0	0

   (a) Categorize the variables as basic and nonbasic, and provide the current values of all the variables.

   (b) Assuming that the problem is of the maximization type, identify the nonbasic variables that have the potential to improve the value of z. If each such variable enters the basic solution, determine the associated leaving variable, if any, and the associated change in z.

   (c) Repeat part (b) assuming that the problem is of the minimization type.

(d) Which nonbasic variable(s) will not cause a change in the value of z when selected to enter the solution?

8. You are given the tableau shown below for a maximization problem.

Basic	x_1	x_2	x_3	x_4	x_5	RHS
z	$-c$	2	0	0	0	10
x_3	-1	a_1	1	0	0	4
x_4	a_2	-4	0	1	0	1
x_5	a_3	3	0	0	1	b

Give conditions on the unknowns a_1, a_2, a_3, b, and c that make the following statements true:

(a) The current solution is optimal.

(b) The current solution is optimal, and there are alternative optimal solutions.

(c) The LP is unbounded (in this part, assume that $b \geq 0$).

9. Suppose we have obtained the tableau shown below for a maximization problem.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	c_1	c_2	0	0	0	0	10
x_3	4	a_1	1	0	a_2	0	b
x_4	-1	-5	0	1	-1	0	2
x_6	a_3	-3	0	0	-4	1	3

State conditions on a_1, a_2, a_3, b, c_1, and c_2 that are required to make the following statements true:

(a) The current solution is optimal, and there are alternative optimal solutions.

(b) The current basic solution is not a basic feasible solution.

(c) The current basic solution is a degenerate bfs.

(d) The current basic solution is feasible, but the LP is unbounded.

(e) The current basic solution is feasible, but the objective function value can be improved by replacing x_6 as a basic variable with x_1.

10. The starting and current tableaux of a given problem are shown below. Find the values of the unknowns a through n.

Starting Tableau

Basic	x_1	x_2	x_3	x_4	x_5	RHS
z	a	1	-3	0	0	0
x_4	b	c	d	1	0	6
x_5	-1	2	e	0	1	1

Current Tableau

Basic	x_1	x_2	x_3	x_4	x_5	RHS
z	0	$-1/3$	j	k	ℓ	n
x_1	g	$2/3$	$2/3$	$2/3$	0	f
x_5	h	i	$-1/3$	$2/3$	1	m

3 Sensitivity Analysis and Duality

Two of the most important topics in linear programming are sensitivity analysis and duality. After studying these important topics, the reader will have an appreciation of the beauty and logic of linear programming.

3.1 Some Important Formulas

In this section, we use our knowledge of matrices to show how an LP's optimal tableau can be expressed in terms of the LP's parameters. The formulas developed in this section are used in our study of sensitivity analysis and duality.

Assume that we are solving a max problem that has m constraints and n variables. Although some of these variables may be slack, excess, or artificial, we choose to label them $x_1, x_2, \cdots, x_n$. Then the LP may be written as

$$\begin{array}{ll} \textbf{max} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \textbf{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_1, x_2, \cdots, x_n \geq 0 \end{array} \tag{3.1}$$

Suppose we have found the optimal solution to (3.1). We define:

Definition 3.1.

1. BV is the set of basic variables in optimal tableau.
2. $\mathbf{X}_{BV}$ is the $m \times 1$ vector of basic variables in optimal tableau.
3. NBV is the set of nonbasic variables in optimal tableau.
4. $\mathbf{X}_{NBV}$ is the $(n-m) \times 1$ vector of nonbasic variables in optimal tableau.
5. $\mathbf{C}_{BV}$ is the $1 \times m$ row vector contains the coefficients of basic variables in the initial tableau.
6. $\mathbf{C}_{NBV}$ is the $1 \times (n-m)$ row vector contains the coefficients of nonbasic variables in the initial tableau.
7. The $m \times m$ matrix $\mathbf{B}$ is the matrix whose jth column is the column of BV_j in the initial tableau.

8. $\mathbf{a}_j$ is the column (in the constraints) of the variable x_j in the initial tableau.
9. $\mathbf{N}$ is the $m \times (n-m)$ matrix whose columns are the coefficients of the nonbasic variables in the initial tableau.
10. $\mathbf{b}$ is the $1 \times m$ column vector contains the right-hand side of the constraints in the initial tableau.

So, the LP in (3.1) can be written as

$$\begin{array}{ll} \textbf{max} & z = \mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{NBV}\mathbf{X}_{NBV} \\ \textbf{s.t.} & \mathbf{B}\mathbf{X}_{BV} + \mathbf{N}\mathbf{X}_{NBV} = \mathbf{b} \\ & \mathbf{X}_{BV}, \mathbf{X}_{NBV} \geq \mathbf{0} \end{array} \tag{3.2}$$

and the initial tableau has the form

Basic	BV	NBV	RHS
z	$\mathbf{C}_{BV}$	$\mathbf{C}_{NBV}$	z value
$\mathbf{X}_{BV}$	$\mathbf{B}_{m\times m}$	$\mathbf{N}_{m\times(n-m)}$	$\mathbf{b}$

Formulas for Computing the Optimal Tableau

1. Right-hand side of optimal tableau's constraints: $\overline{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$.
2. Right-hand side of optimal row 0: $\overline{z} = \mathbf{C}_{BV}\left(\mathbf{B}^{-1}\mathbf{b}\right) = \mathbf{C}_{BV}\overline{\mathbf{b}}$.
3. x_j columns in optimal tableau's constraints: $\overline{\mathbf{a}_j} = \mathbf{B}^{-1}\mathbf{a}_j$. In general, $\mathbf{X}_{NBV}$ columns in optimal tableau's constraints: $\overline{\mathbf{N}} = \mathbf{B}^{-1}\mathbf{N}$ and $\mathbf{X}_{NBV}$ column in optimal tableau's constraints $= \mathbf{I}_m$.
4. (a) $\overline{c_j} = \mathbf{C}_{BV}\left(\mathbf{B}^{-1}\mathbf{a}_j\right) - c_j = \mathbf{C}_{BV}\overline{\mathbf{a}_j} - c_j$.
   (b) $\overline{c}_{s_i} = i$th element in $\mathbf{C}_{BV}\mathbf{B}^{-1}$.
   (c) $\overline{c}_{e_i} = -i$th element in $\mathbf{C}_{BV}\mathbf{B}^{-1}$.
   (d) In maximization LP, $\overline{c}_{a_i} = i$th element in $\mathbf{C}_{BV}\mathbf{B}^{-1} + M$.
   (e) In minimization LP, $\overline{c}_{a_i} = i$th element in $\mathbf{C}_{BV}\mathbf{B}^{-1} - M$.
   (f) **In general**, $\overline{\mathbf{C}}_{NBV} = \mathbf{C}_{BV}\left(\mathbf{B}^{-1}N\right) - \mathbf{C}_{NBV} = \mathbf{C}_{BV}\overline{\mathbf{N}} - \mathbf{C}_{NBV}$.

The optimal tableau then has the form:

Basic	BV	NBV	RHS
z	$\mathbf{0}$	$\mathbf{C}_{BV}\left(\mathbf{B}^{-1}N\right) - \mathbf{C}_{NBV}$	$\mathbf{C}_{BV}\left(\mathbf{B}^{-1}\mathbf{b}\right)$
$\mathbf{X}_{BV}$	$\mathbf{I}_{m\times m}$	$\mathbf{B}^{-1}\mathbf{N}_{m\times(n-m)}$	$\mathbf{B}^{-1}\mathbf{b}$

Formulas Derivation:

1. To expressing the constraints in any tableau in terms of $\mathbf{B}^{-1}$ and the original LP, we observe that

   $$\mathbf{B}\mathbf{X}_{BV} + \mathbf{N}\mathbf{X}_{NBV} = \mathbf{b}$$

   Multiplying both sides from the left by $\mathbf{B}^{-1}$, we obtain

   $$\mathbf{B}^{-1}\mathbf{B}\mathbf{X}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$

   which implies that

   $$\mathbf{X}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$

2. To determining the optimal tableau's Row 0 in terms of the initial LP, we rewrite the original objective function, $z = \mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{NBV}\mathbf{X}_{NBV}$ as

   $$z - \mathbf{C}_{BV}\mathbf{X}_{BV} - \mathbf{C}_{NBV}\mathbf{X}_{NBV} = 0 \tag{3.3}$$

   Also, we multiply the constraints expressed in the form $\mathbf{B}\mathbf{X}_{BV}+\mathbf{N}\mathbf{X}_{NBV} = \mathbf{b}$ through by the vector $\mathbf{C}_{BV}\mathbf{B}^{-1}$ to obtain

   $$\mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N}\mathbf{X}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{b} \tag{3.4}$$

   By adding equation (3.3) to equation (3.4), we obtain

   $$z + \left(\mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV}\right)\mathbf{X}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{b}$$

Example 3.1. For the following LP, the optimal basis is $BV = \{x_2, s_2\}$. Compute the optimal tableau.

$$\begin{array}{ll} \textbf{max} & z = x_1 + 4x_2 \\ \textbf{s.t.} & x_1 + 2x_2 \le 6 \\ & 2x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{array}$$

Solution: After adding slack variables s_1 and s_2, the LP in standard form

$$\begin{array}{ll} \textbf{max} & z = x_1 + 4x_2 \\ \textbf{s.t.} & x_1 + 2x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{array}$$

Since $BV = \{x_2, s_2\}$ and $NBV = \{x_1, s_1\}$, then

$$\mathbf{C}_{BV} = \begin{bmatrix} 4 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_{NBV} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \quad , \quad \mathbf{N} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

So, the optimal tableau entries are

$$\begin{aligned}
\overline{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} &= \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\
\overline{\mathbf{N}} = \mathbf{B}^{-1}\mathbf{N} &= \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} \\
\overline{\mathbf{C}}_{NBV} = \mathbf{C}_{BV}\overline{\mathbf{N}} - \mathbf{C}_{NBV} &= \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \\
\overline{z} = \mathbf{C}_{BV}\overline{\mathbf{b}} &= \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 12.
\end{aligned}$$

and the optimal tableau is

Basic	x_1	x_2	s_1	s_2	RHS
z	1	0	2	0	12
x_2	1/2	1	1/2	0	3
s_2	3/2	0	−1/2	1	5

Example 3.2. For the following LP, the optimal basis is $BV = \{x_2, x_4\}$. Compute the optimal tableau.

$$\begin{array}{ll}
\textbf{max} & z = x_1 + 4x_2 + 7x_3 + 5x_4 \\
\textbf{s.t.} & 2x_1 + x_2 + 2x_3 + 4x_4 = 10 \\
 & 3x_1 - x_2 - 2x_3 + 6x_4 = 5 \\
 & x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

Solution: Note that the constraints are in equation form, and no need to add artificial variables here (we do not solve by simplex). Since $BV = \{x_2, x_4\}$ and $NBV = \{x_1, x_3\}$, then

$$\begin{array}{llllll}
\mathbf{C}_{BV} = \begin{bmatrix} 4 & 5 \end{bmatrix} & , & \mathbf{C}_{NBV} = \begin{bmatrix} 1 & 7 \end{bmatrix} & , & \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \\
\mathbf{B} = \begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix} & , & \mathbf{B}^{-1} = \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} & , & \mathbf{N} = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix}
\end{array}$$

So, the optimal tableau entries are

$$\begin{aligned}
\overline{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} &= \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3/2 \end{bmatrix} \\
\overline{\mathbf{N}} = \mathbf{B}^{-1}\mathbf{N} &= \begin{bmatrix} 3/5 & -2/5 \\ 1/10 & 1/10 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix} \\
\overline{\mathbf{C}}_{NBV} = \mathbf{C}_{BV}\overline{\mathbf{N}} - \mathbf{C}_{NBV} &= \begin{bmatrix} 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 7 \end{bmatrix} = \begin{bmatrix} 3/2 & 1 \end{bmatrix}
\end{aligned}$$

$$\overline{z} = \mathbf{C}_{BV}\overline{\mathbf{b}} = \begin{bmatrix} 4 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3/2 \end{bmatrix} = \frac{47}{2}.$$

and the optimal tableau is

Basic	x_1	x_2	x_3	x_4	RHS
z	$3/2$	0	1	0	$47/2$
x_2	0	1	2	0	4
x_4	$1/2$	0	0	1	$3/2$

Note 11. We have used the formulas of this section to create an LP's optimal tableau, but they can also be used to create the tableau for any set of basic variables.

Exercise 3.1.

1. For the following LP, x_1 and x_2 are basic variables in the optimal tableau. Use the formulas of matrices to determine the optimal tableau.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + x_2 \\ \textbf{s.t.} & 2x_1 - x_2 \leq 2 \\ & -x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

2. For the following LP, x_2 and s_1 are basic variables in the optimal tableau. Use the formulas of matrices to determine the optimal tableau.

$$\begin{array}{ll} \textbf{max} & z = -x_1 + x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

3. Consider the following LP model:

$$\begin{array}{ll} \textbf{max} & z = 5x_1 + 2x_2 + 3x_3 \\ \textbf{s.t.} & x_1 + 5x_2 + 2x_3 \leq b_1 \\ & x_1 - 5x_2 - 6x_3 \leq b_2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The following optimal tableau corresponds to specific values of b_1 and b_2:

Basic	x_1	x_2	x_3	s_1	s_2	RHS
z	0	a	7	d	e	150
x_1	1	b	2	1	0	30
s_2	0	c	-8	-1	1	10

Determine the elements a, b, c, d, e, b_1 and b_2.

4. For the following LP, x_1 and x_2 are basic variables in the optimal tableau. Determine the optimal tableau using the laws of matrices.

$$\begin{array}{ll} \textbf{min} & w = 50x_1 + 100x_2 \\ \textbf{s.t.} & 7x_1 + 2x_2 \geq 28 \\ & 2x_1 + 12x_2 \geq 24 \\ & x_1, x_2 \geq 0 \end{array}$$

3.2 Sensitivity Analysis

We now explore how changes in an LP's parameters (objective function coefficients, right-hand sides, and technological coefficients) change the optimal solution. The study of how an LP's optimal solution depends on its parameters is called sensitivity analysis. Our discussion focuses on maximization problems and relies heavily on the formulas of Section 3.1. (The modifications for min problems are straightforward; see Exercise 3.2 at the end of this section.)

As in Section 3.1, we let BV be the set of basic variables in the optimal tableau. Given a change (or changes) in an LP, we want to determine whether BV remains optimal. The mechanics of sensitivity analysis hinge on the following important observation. From Chapter 2, we know that a simplex tableau (for a max problem) for a set of basic variables BV is optimal if and only if each constraint has a nonnegative right-hand side and each variable has a nonnegative coefficient in row 0. This implies that whether a tableau is feasible and optimal depends only on the right-hand sides of the constraints and on the coefficients of each variable in row 0.

Suppose we have solved an LP and have found that BV is an optimal basis. We can use the following procedure to determine if any change in the LP will cause BV to be no longer optimal.

Step 1: Using the formulas of Section 3.1, determine how changes in the LP's parameters change the right-hand side and row 0 of the optimal tableau (the tableau having BV as the set of basic variables).

Step 2: If each variable in row 0 has a non-negative coefficient and each constraint has a nonnegative right-hand side, then BV is still optimal. Otherwise, BV is no longer optimal.

We will discuss how 6 types of changes on LP's parameters change the optimal solution:

1. **Changing the objective function coefficient of a nonbasic variable:** If the objective function coefficient for a nonbasic variable x_j is changed, the current basis remains optimal if $\overline{c}_j \geq 0$. If $\overline{c}_j < 0$, then the current basis is no longer optimal, and x_j will be a basic variable in the new optimal solution.

2. **Changing the objective function coefficient of a basic variable:** If the objective function coefficient of a basic variable x_j is changed, then the current basis remains optimal if the coefficient of every variable in row 0 of the BV tableau remains nonnegative. If any variable in row 0 has a negative coefficient, then the current basis is no longer optimal.

3. **Changing the right-hand side of a constraint:** If the right-hand side of a constraint is changed, then the current basis remains optimal if the right-hand side of each constraint in the tableau remains nonnegative. If the right-hand side of any constraint is negative, then the current basis is infeasible, and a new optimal solution must be found.

4. **Changing a column of a nonbasic variable:** If the column of a nonbasic variable x_j is changed, then the current basis remains optimal if $\overline{c}_j \geq 0$. If $\overline{c}_j < 0$, then the current basis is no longer optimal and x_j will be a basic variable in the new optimal solution. If the column of a basic variable is changed, then it is usually difficult to determine whether the current basis remains optimal. This is because the change may affect both $\mathbf{B}$ and $\mathbf{C}_{BV}$ and thus the entire row 0 and the entire right-hand side of the optimal tableau. As always, the current basis would remain optimal if and only if each variable has a nonnegative coefficient in row 0 and each constraint has a nonnegative right-hand side.

5. **Adding a new variable:** If a new column (corresponding to a variable x_j) is added to an LP, then the current basis remains optimal if $\overline{c}_j \geq 0$. If $\overline{c}_j < 0$, then the current basis is no longer optimal and x_j will be a basic variable in the new optimal solution.

6. **Adding a new constraint.** (see Section 3.7)

When applying the techniques of this section to a minimization problem, just remember that a tableau is optimal if and only if each variable has a nonpositive coefficient in row 0 and the right-hand side of each constraint is nonnegative.

Example 3.3. Consider the following LP:

$$\begin{array}{ll}\textbf{max} & z = 60x_1 + 30x_2 + 20x_3 \\ \textbf{s.t.} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + \frac{3}{2}x_3 \leq 20 \\ & 2x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

After adding slack variables s_1, s_2, and s_3, the optimal tableau is:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	RHS
z	0	5	0	0	10	10	280
s_1	0	-2	0	1	2	-8	24
x_3	0	-2	1	0	2	-4	8
x_1	1	$5/4$	0	0	$-1/2$	$3/2$	2

1. Suppose we change the objective function coefficient of x_2 from 30 to $30+\Delta$. For what values of Δ will the current set of basic variables remain optimal ?

   Solution: From the optimal tableau we know that $BV = \{s_1, x_3, x_1\}$ and $NBV = \{x_2, s_2, s_3\}$, then

$$\mathbf{C}_{BV} = \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \quad , \quad \mathbf{C}_{NBV} = \begin{bmatrix} 30 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 3/2 & 4 \\ 0 & 1/2 & 2 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \quad , \quad \mathbf{N} = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix}$$

   Because x_2 is a nonbasic variable, $\mathbf{C}_{BV}$ has not changed. Thus, BV will remain optimal if

$$\begin{aligned} \overline{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 30+\Delta & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5-\Delta & 10 & 10 \end{bmatrix} \geq \mathbf{0} \\ \therefore \quad & \Delta \leq 5 \end{aligned}$$

2. Suppose we change the objective function coefficient of x_1 from 60 to $60+\Delta$. For what values of Δ will the current set of basic variables remain optimal ?

   Solution: The BV will remain optimal if

$$\begin{aligned} \overline{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= \begin{bmatrix} 0 & 20 & 60+\Delta \end{bmatrix} \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5+\frac{5}{4}\Delta & 10-\frac{1}{2}\Delta & 10+\frac{3}{2}\Delta \end{bmatrix} \geq \mathbf{0} \\ \therefore \quad & \Delta \in [-4, 20] \end{aligned}$$

3. Suppose we change the right-hand-side of the second constraint from 20 to $20 + \Delta$. For what values of Δ will the current set of basic variables remain optimal ?

   Solution: The BV will remain optimal if

$$\begin{aligned} \overline{\mathbf{b}} &= \mathbf{B}^{-1}\mathbf{b} \\ &= \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 48 \\ 20+\Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24+2\Delta \\ 8+2\Delta \\ 2-\frac{1}{2}\Delta \end{bmatrix} \geq \mathbf{0} \\ \therefore \quad & \Delta \in [-4, 4] \end{aligned}$$

4. Suppose we change the elements of the column for x_2 from $\begin{bmatrix} 30 \\ \hline 6 \\ 2 \\ 3/2 \end{bmatrix}$ to $\begin{bmatrix} 43 \\ \hline 5 \\ 2 \\ 2 \end{bmatrix}$. Would this change the optimal solution to the problem ?

   Solution: Thus, BV will remain optimal if $\overline{\mathbf{C}}_{NBV} \geq \mathbf{0}$. But

$$\begin{aligned} \overline{\mathbf{C}}_{NBV} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} \\ &= \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 43 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 10 & 10 \end{bmatrix} \ngeq \mathbf{0} \end{aligned}$$

   $\therefore$ The current basis is no longer optimal.

5. Suppose we add new activity x_4 to the problem, and we add the column $\begin{bmatrix} 15 \\ \hline 1 \\ 1 \\ 1 \end{bmatrix}$ for x_4 to the problem. How will the addition of the new activity change the optimal tableau?

   Solution: Because

$$\begin{aligned} \bar{c}_{x_4} &= \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{a}_4 - c_{x_4} \\ &= \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 \\ &= 5 \geq 0 \end{aligned}$$

   $\therefore$ The current basis is still optimal.

Exercise 3.2.

1. For the following LP, x_1 and x_2 are basic variables in the optimal tableau. For what values of α, the current basis remains optimal?

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + x_2 \\ \textbf{s.t.} & 2x_1 - x_2 \leq 2 \\ & -x_1 + x_2 \leq \alpha \\ & x_1, x_2 \geq 0 \end{array}$$

2. If the basic variables in the optimal solution of the following LP is $BV = \{x_1, x_3\}$.

$$\begin{array}{ll} \textbf{max} & z = 4x_1 - 3x_2 + 2x_3 \\ \textbf{s.t.} & x_1 + 2x_2 - x_3 \leq 5 \\ & x_1 + 2x_2 + x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   (a) Use laws of matrices to find the optimal $z-$value.

   (b) If we change the right-hand side of the second constraint from 7 to $7 + \Delta$, then for what values of Δ the current basis remains optimal?

   (c) Suppose we change the coefficient of x_2 in the objective function from -3 to $-3 + \Delta$, then for what values of Δ the current basis remains optimal?

   (d) If the coefficients of x_1 in the objective function is changed from 4 to 1, does the current basis remain optimal?

3.3 Finding the Dual of an LP

Every linear program (LP) has a corresponding *dual* LP, and understanding the relationship between them is crucial for advanced topics in linear and nonlinear programming, offering valuable economic and sensitivity analysis insights. The original LP is called the *primal*, and the optimal solution of one problem directly determines the optimal solution of the other. If the primal is a maximization problem, its dual will be a minimization problem, and vice versa. We begin by explaining how to construct the dual of a *normal max problem*, where all variables are nonnegative and constraints are of the form ($\leq$).

$$\begin{array}{ll} \textbf{max} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \textbf{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \cdots, x_n \geq 0 \end{array} \tag{3.5}$$

The dual of a normal max problem such as (3.5) is defined to be

$$\begin{array}{ll}
\textbf{min} & w = b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\textbf{s.t.} & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
 & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
 & \vdots \\
 & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
 & y_1, y_2, \cdots, y_m \geq 0
\end{array} \tag{3.6}$$

A min problem such as (3.6) that has all ($\geq$) constraints and all variables nonnegative is called a *normal min problem*. If the primal is a normal min problem such as (3.6), then we define the dual of (3.6) to be (3.5).

Finding the Dual:

1. If the primal is a **maximization** problem, the dual will be a **minimization** problem, and vice versa. The dual of the dual problem yields the original problem.
2. A *dual variable* is defined for each *primal constraint* equation. Also, a *dual constraint* is defined for each *primal variable*.
3. The column coefficients in a constraint of a primal variable defines the left-hand side coefficients of the dual constraint and its objective coefficient defines the right-hand side of that constraint. The objective coefficients of the dual equal the right-hand side of the primal constraint.
4. The variables and constraints in the primal and dual problems are related as follows.

(max)	$\Longleftrightarrow$	**(min)**
Constraint Sign		**Variable Sign**
$\geq$	$\Longleftrightarrow$	≤ 0
$\leq$	$\Longleftrightarrow$	≥ 0
$=$	$\Longleftrightarrow$	u.r.s
Variable Sign		**Constraint Sign**
≤ 0	$\Longleftrightarrow$	$\leq$
≥ 0	$\Longleftrightarrow$	$\geq$
u.r.s	$\Longleftrightarrow$	$=$

Example 3.4. Find the dual of the following LPs:

1. **min** $w = x_1 + 2x_2$
   s.t. $2x_1 + x_2 \geq 4$
   $x_1 - 2x_2 \geq 8$
   $x_1, x_2 \geq 0$

2. **max** $z = 5x_1 + 12x_2 + 4x_3$
   s.t. $x_1 + 2x_2 + x_3 \leq 10$
   $2x_1 - x_2 + x_3 = 8$
   $x_1, x_2, x_3 \geq 0$

3. **min** $w = 15x_1 + 12x_2$
   s.t. $x_1 + 2x_2 \geq 3$
   $x_1 - 4x_2 \leq 5$
   $x_1, x_2 \geq 0$

4. **max** $z = 5x_1 + 6x_2$
   s.t. $x_1 + 2x_2 = 5$
   $-x_1 + 5x_2 \geq 3$
   $4x_1 + 7x_2 \leq 8$
   x_1 urs, $x_2 \geq 0$

Solution:

1. **max** $z = 4y_1 + 8y_2$
   s.t. $2y_1 + y_2 \leq 1$
   $y_1 - 2y_2 \leq 2$
   $y_1, y_2 \geq 0$

2. **min** $w = 10y_1 + 8y_2$
   s.t. $y_1 + 2y_2 \geq 5$
   $2y_1 - y_2 \geq 12$
   $y_1 + y_2 \geq 4$
   $y_1 \geq 0$, y_2 urs

3. **max** $z = 3y_1 + 5y_2$
   s.t. $y_1 + y_2 \leq 15$
   $2y_1 - 4y_2 \leq 12$
   $y_1 \geq, y_2 \leq 0$

4. **min** $w = 5y_1 + 3y_2 + 8y_3$
   s.t. $y_1 - y_2 + 4y_3 = 5$
   $2y_1 + 5y_2 + 7y_3 \geq 6$
   y_1 urs, $y_2 \leq 0$, $y_3 \geq 0$

Exercise 3.3. Find the dual of the following LPs:

1. **max** $z = 2x_1 + x_2$
   s.t. $-x_1 + x_2 \leq 1$
   $x_1 + x_2 \leq 3$
   $x_1 - 2x_2 \leq 4$
   $x_1, x_2 \geq 0$

2. **min** $w = y_1 - y_2$
   s.t. $2y_1 + y_2 \geq 4$
   $y_1 + y_2 \geq 1$
   $y_1 + 2y_2 \geq 3$
   $y_1, y_2 \geq 0$

3. **max** $z = 4x_1 - x_2 + 2x_3$
   s.t. $x_1 + x_2 \leq 5$
   $2x_1 + x_2 \leq 7$
   $2x_2 + x_3 \geq 6$
   $x_1 + x_3 = 4$
   $x_1 \geq 0$, x_2, x_3 urs

4. **min** $w = 4y_1 + 2y_2 - y_3$
   s.t. $y_1 + 2y_2 \leq 6$
   $y_1 - y_2 + 2y_3 = 8$
   $y_1, y_2 \geq$, y_3 urs

3.4 The Dual Theorem and its Consequences

In this section, we discuss one of the most important results in linear programming: the Dual Theorem. In essence, the Dual Theorem states that the primal and dual have equal optimal objective function values (if the problems have optimal solutions).

If we choose any feasible solution to the max LP and any feasible solution to the min LP (one is primal and the other is dual), the value for the min LP feasible solution will be at least as large as the value for the max LP feasible solution. This result is formally stated in Lemma 3.1.

Lemma 3.1. *The objective values in a pair of primal-dual problems must satisfy the following relationships:*

1. *For any pair of feasible primal and dual solutions,*

$$\begin{pmatrix} \textit{objective value} \\ \textit{in } \textbf{MAX} \textit{ LP} \end{pmatrix} \leq \begin{pmatrix} \textit{objective value} \\ \textit{in } \textbf{MIN} \textit{ LP} \end{pmatrix}$$

2. *At the optimum solution for both problems,*

$$\begin{pmatrix} \textit{objective value} \\ \textit{in } \textbf{MAX} \textit{ LP} \end{pmatrix} = \begin{pmatrix} \textit{objective value} \\ \textit{in } \textbf{MIN} \textit{ LP} \end{pmatrix}$$

Proof. Consider the primal LP

$$\begin{array}{ll} \textbf{max} & z = \mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{NBV}\mathbf{X}_{NBV} \\ \textbf{s.t.} & \mathbf{A}\mathbf{X}_{NBV} + \mathbf{I}\mathbf{X}_{BV} = \mathbf{b} \\ & \mathbf{X}_{BV}, \mathbf{X}_{NBV} \geq \mathbf{0} \end{array} \tag{3.7}$$

Then, the dual LP will be

$$\begin{array}{ll} \textbf{min} & w = \mathbf{Y}\mathbf{b} \\ \textbf{s.t.} & \mathbf{Y}\mathbf{A} \geq \mathbf{C}_{NBV} \\ & \mathbf{Y}\mathbf{I} \geq \mathbf{C}_{BV} \\ & \mathbf{Y} \text{ urs} \end{array} \tag{3.8}$$

Multiply the constraint in (3.7) by $\mathbf{Y}$ from the left to obtain:

$$\mathbf{YAX}_{NBV} + \mathbf{YIX}_{BV} = \mathbf{Yb} = w.$$

Also, multiply the first constraint in (3.8) by $\mathbf{X}_{BV}$ from the right, and the second constraint by $\mathbf{X}_{NBV}$ from the right, to obtain:

$$\mathbf{YAX}_{BV} \geq \mathbf{C}_{NBV}\mathbf{X}_{BV}$$
$$\mathbf{YIX}_{NBV} \geq \mathbf{C}_{BV}\mathbf{X}_{NBV}$$

By adding the two inequalities above, we have

$$\mathbf{YAX}_{NBV} + \mathbf{YIX}_{BV} \geq \mathbf{C}_{BV}\mathbf{X}_{BV} + \mathbf{C}_{NBV}\mathbf{X}_{NBV}$$
$$\underbrace{w}_{\textbf{min}} \geq \underbrace{z}_{\textbf{max}}$$

■

Example 3.5. Consider the following pair of primal and dual problems.

Primal

$$\begin{array}{llllll} \textbf{min} & w = 5x_1 + 2x_2 & & & & \\ \textbf{s.t.} & & x_1 - & x_2 & \geq 3 \\ & 2 & x_1 + & 3x_2 & \geq 5 \\ & x_1,\ x_2 \geq 0 & & & \end{array}$$

Feasible Solution:
$x_1 = 4,\ x_2 = 1$
Objective Function:
$w = 22$

Dual

$$\begin{array}{llllll} \textbf{max} & z = 3y_1 + 5y_2 & & & & \\ \textbf{s.t.} & & y_1 + & 2y_2 & \leq 5 \\ & - & y_1 + & 3y_2 & \leq 2 \\ & y_1,\ y_2 \geq 0 & & & \end{array}$$

Feasible Solution (Optimal):
$y_1 = 5,\ y_2 = 0$
Objective Function:
$z = 15$

Theorem 3.2. The Dual Theorem.[a]
Suppose BV is an optimal basis for the primal. Then $\mathbf{y} = \mathbf{C}_{BV}\mathbf{B}^{-1}$ is an optimal solution to the dual. Also, $\overline{z} = \overline{w}$.

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[a] **For proof see:** Wayne L. Winston, Munirpallam Venkataramanan. **Introduction to Mathematical Programming**. Thomson Learning; 4th edition (2002)

Example 3.6. The optimal solution of the following LP is $z = 9$ when $x_1 = 1$ and $x_2 = 6$. Find its dual problem, then find the solution for the dual problem.

$$\begin{array}{ll} \textbf{max} & z = 3x_1 + x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \leq 8 \\ & 4x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: Since, in the optimal solution, $BV = \{x_1, x_2\}$ then

$$\mathbf{C}_{BV} = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} \quad , \quad \mathbf{B}^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 2 & -1 \end{bmatrix}$$

Hence, $\overline{\mathbf{y}} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \mathbf{C}_{BV}\mathbf{B}^{-1} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$ and $\overline{w} = \overline{z} = 9$. Note that the dual LP is

$$\begin{array}{ll} \textbf{min} & w = 8y_1 + 10y_2 \\ \textbf{s.t.} & 2y_1 + 4y_2 \geq 3 \\ & y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{array}$$

How to Read the Optimal Dual Solution from Row 0 of the Optimal Tableau ?

Constraint i Sign	Optimal y_i Value	Problem Type
$\leq$	Coefficient of s_i	Max or Min
$\geq$	$-1\times$ Coefficient of e_i	Max or Min
$=$	Coefficient of $a_i - M$	Max
$=$	Coefficient of $a_i + M$	Min

In general,

$$\left\{\begin{array}{c}\text{The optimal}\\ \text{value of the dual}\\ \text{variable } y_i\end{array}\right\} = \left\{\begin{array}{c}\text{Optimal primal } z\text{−coefficient of the starting variable } d_i\\ +\\ \text{Original objective function coefficient of } d_i\end{array}\right\}$$

Example 3.7. Consider the following LP.

$$\begin{array}{ll}\textbf{max} & z = -2x_1 - x_2 + x_3\\ \textbf{s.t.} & x_1 + x_2 + x_3 \leq 3\\ & x_2 + x_3 \geq 2\\ & x_1 + x_3 = 1\\ & x_1, x_2, x_3 \geq 0\end{array}$$

1. Find the dual of this LP.

   Solution: The dual LP is

$$\begin{array}{ll}\textbf{min} & w = 3y_1 + 2y_2 + y_3\\ \textbf{s.t.} & y_1 + y_3 \geq -2\\ & y_1 + y_2 \geq -1\\ & y_1 + y_2 + y_3 \geq 1\\ & y_1 \geq 0,\ y_2 \leq 0,\ y_3 \text{ urs}\end{array}$$

2. After adding slack variable s_1, subtracting excess variable e_2, and adding artificial variables a_2 and a_3, the **Row 0** of the LP's optimal tableau is found to be

$$z + 4x_1 + e_2 + (M-1)a_2 + (M+2)a_3 = 0.$$

   Find the optimal solution of the dual problem.

   Solution: The starting primal variables s_1, a_2 and a_3 uniquely correspond to the dual variables y_1, y_2 and y_3, respectively. Thus, the optimum dual solution is $\overline{w} = \overline{z} = 0$, and

Using the Table	**Using the General Formula**
$y_1 = 0$	$y_1 = 0 + 0 = 0$
$y_2 = -1$	$y_2 = -M + (M-1) = -1$
$y_3 = (M+2) - M = 2$	$y_3 = -M + (M+2) = 2$

3. Suppose we change the right-hand side of the third constraint from 1 to 2, what is the change on the $z-$value will be? Assume the current basis remains optimal.

   Solution: $z_{\text{new}} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\mathbf{y}}\mathbf{b}_{\text{new}} = \begin{bmatrix}0 & -1 & 2\end{bmatrix}\begin{bmatrix}3\\2\\2\end{bmatrix} = 2.$

4. Repeat part (3) if we change the right-hand side of the second constraint from 2 to 5.

   Solution: $z_{\text{new}} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\mathbf{y}}\mathbf{b}_{\text{new}} = \begin{bmatrix}0 & -1 & 2\end{bmatrix}\begin{bmatrix}3\\5\\1\end{bmatrix} = -3.$

Corollary 3.3. *The primal problem is infeasible if and only if the normal form of the dual problem is unbounded (and vice versa).*

Note 12. With regard to the primal and dual linear programming problems, exactly one of the following statements is true:

1. Both possess optimal solutions.
2. One problem has an unbounded optimal objective value, in which case the other problem must be infeasible.
3. Both problems are infeasible.

From this note we see that duality is not completely symmetric. The best we can say is that (here optimal means having a finite optimum, and unbounded means having an unbounded optimal objective value):

Primal **Optimal** $\Leftrightarrow$ Dual **Optimal**
Primal (Dual) **Unbounded** $\Rightarrow$ Dual (Primal) **Infeasible**
Primal (Dual) **Infeasible** $\Rightarrow$ Dual (Primal) **Unbounded** or **Infeasible**
Primal (Dual) **Infeasible** $\Leftrightarrow$ Dual (Primal) **Unbounded in normal form**

Note 13. The relationship between degeneracy and multiplicity of the primal and the dual optimal solutions is formulated in Theorem 3.4. Recall that *degeneracy* and *multiplicity* always refer to LP models with inequality constraints, and that *degeneracy* is defined for basic feasible solutions. In this theorem, the term *nondegenerate* in the expression "multiple and nondegenerate" means that there are multiple optimal solutions, and that there exists an optimal basic feasible solution that is nondegenerate.

Theorem 3.4. Duality relationships between degeneracy and multiplicity.
For any pair of primal and dual standard LP-models where both have optimal solutions, the following implications hold:

Primal optimal solution		**Dual optimal solution**
Multiple	$\Rightarrow$	*Degenerate*
Unique and nondeg.	$\Rightarrow$	*Unique and nondeg.*
Multiple and nondeg.	$\Rightarrow$	*Unique and degenerate*
Unique and degenerate	$\Rightarrow$	*Multiple*

Exercise 3.4.

1. Find the optimal value of the objective function for the following LP using its dual. (Do NOT solve the dual using the simplex algorithm)

$$\begin{aligned} \textbf{min} \quad & w = 10y_1 + 4y_2 + 5y_3 \\ \textbf{s.t.} \quad & 5y_1 - 7y_2 + 3y_3 \ \geq 50 \\ & y_1,\ y_2,\ y_3 \geq 0 \end{aligned}$$

2. Consider the following LP.

$$\begin{aligned} \textbf{max} \quad & z = 2x_1 + 4x_2 + 4x_3 - 3x_4 \\ \textbf{s.t.} \quad & x_1 + \ \ x_2 + x_3 \quad\ = 4 \\ & \qquad 4x_2 + \quad x_4 = 8 \\ & x_1,\ x_2,\ x_3,\ x_4 \geq 0 \end{aligned}$$

   (a) Write the associated dual problem.
   (b) Show that the basic solution x_1 and x_2 is not optimal.
   (c) Using x_3 and x_4 as starting variables, the optimal tableau is given below. Determine the dual optimal solution in **TWO** ways, using the tableau.

Basic	x_1	x_2	x_3	x_4	RHS
z	2	0	0	3	16
x_3	$3/4$	0	1	$-1/4$	2
x_2	$1/4$	1	0	$1/4$	2

3. For the following LP,

$$\begin{aligned} \textbf{max} \quad & z = -x_1 + 5x_2 \\ \textbf{s.t.} \quad & \ \ x_1 + 2x_2 \leq 0.5 \\ & -x_1 + 3x_2 \leq 0.5 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

row 0 of the optimal tableau is $z + 0.4s_1 + 1.4s_2 = \boxed{?}$. Determine the optimal $z-$value for the given LP.

4. Consider the following linear programming problem:

$$\begin{aligned} \textbf{max} \quad & z = 4x_1 + x_2 \\ \textbf{s.t.} \quad & 3x_1 + 2x_2 \leq 6 \\ & 6x_1 + 3x_2 \leq 10 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Suppose that in solving this problem, row 0 of the optimal tableau is found to be $z + 2x_2 + s_2 = \frac{20}{3}$. Use the Dual Theorem to prove that the computations must be incorrect.

5. Consider the following LP:

$$\begin{aligned} \textbf{max} \quad & z = 5x_1 + 2x_2 + 3x_3 \\ \textbf{s.t.} \quad & x_1 + 5x_2 + 2x_3 \ = 15 \\ & x_1 - 5x_2 - 6x_3 \ \leq 20 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Given that the artificial variable a_1 and the slack variable s_2 form the starting basic variables and that M was set equal to 100 when solving the problem, the optimal tableau is given as:

Basic	x_1	x_2	x_3	a_1	s_2	RHS
z	0	23	7	105	0	75
x_1	1	5	2	1	0	15
s_2	0	-10	-8	-1	1	5

Write the associated dual problem, and determine its optimal solution in two ways.

3.5 Shadow Prices

It is often important for managers to determine how a change in a constraint's right-hand side changes the LP's optimal $z-$value.

Definition 3.2. The shadow price of the ith constraint is the amount by which the optimal value of the objective function is improved (improved means increased in max LP and decreased in min LP) if we increase b_i (the RHS of that constraint) by 1 (from b_i to $b_i + 1$).

Note 14.

1. The previous definition assumes that after the RHS of constraint i has been changed to $b_i + 1$, the current basis remains optimal.

2. The shadow price of the ith constraint of a max LP is the optimal value of the ith dual variable $\overline{y}_i$. Also, the shadow price of the ith constraint of a min LP is $-1\times$ the optimal value of the ith dual variable $\overline{y}_i$

$$\begin{aligned} \overline{z}_{\text{new}} &= \mathbf{Yb} = \begin{bmatrix} \overline{y}_1 & \cdots & \overline{y}_i & \cdots & \overline{y}_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_i + 1 \\ \vdots \\ b_m \end{bmatrix} \\ &= \overline{y}_1 b_1 + \cdots + \overline{y}_i \left(b_i + 1\right) + \cdots + \overline{y}_m b_m \\ &= \left(\overline{y}_1 b_1 + \cdots + \overline{y}_i b_i + \cdots + \overline{y}_m b_m\right) + \overline{y}_i \\ &= \overline{z}_{\text{old}} + y_i \\ \therefore \quad \overline{z}_{\text{new}} - \overline{z}_{\text{old}} &= \overline{y}_i \end{aligned}$$

3. In max LP, the shadow price for a $(\leq)$ constraint is nonnegative, for a $(\geq)$ is nonpositive, and for $(=)$ is urs. Also, in min LP, the shadow price for a $(\leq)$ constraint is nonpositive, for a $(\geq)$ is nonnegative, and for $(=)$ is urs.

4. In general, if the RHS of the ith constraint is increased by an amount Δb_i, then

$$\begin{aligned} \overline{z}_{\text{new}} &= \overline{z}_{\text{old}} + \left(\Delta b_i\right) \overline{y}_i \quad && \text{if the LP is max} \\ \overline{z}_{\text{new}} &= \overline{z}_{\text{old}} - \left(\Delta b_i\right) \overline{y}_i \quad && \text{if the LP is min} \end{aligned}$$

Example 3.8. Consider the following LP.

$$\begin{aligned} \textbf{max} \quad & z = 15x_1 + 25x_2 \\ \textbf{s.t.} \quad & 3x_1 + 4x_2 \leq 100 \\ & 2x_1 + 3x_2 \leq 70 \\ & x_1 + 2x_2 \leq 30 \\ & x_2 \geq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

The optimal solution of the problem is $z = 435$, when $x_1 = 24$ and $x_2 = 3$, where the Row 0 in the optimal tableau (after adding slack variables s_1, s_2, s_3 to the first three constraints respectively and subtracting excess variable e_4 from the last constraint then adding to it an artificial variable a_4) is

$$z + 15s_3 + 5e_4 + (M - 5)a_4 = 435.$$

1. Find the shadow price of each constraint.

   Solution: The shadow price of each constraint is the optimal value of the corresponding dual variable of each constraint. So,

   $$y_1 = 0 \quad , \quad y_2 = 0 \quad , \quad y_3 = 15 \quad , \quad y_4 = -5$$

2. Assuming the current basis remains optimal, what would the change on the $z-$value be if the RHS of the

   (a) 3rd constraint were changed from 30 to 35 ?

   Solution: $\bar{z}_{\text{new}} = \bar{\mathbf{y}}\mathbf{b} = \begin{bmatrix} 0 & 0 & 15 & -5 \end{bmatrix} \begin{bmatrix} 100 \\ 70 \\ 35 \\ 3 \end{bmatrix} = 510.$

   (b) 4th constraint were changed from 3 to 2 ?

   Solution: $\bar{z}_{\text{new}} = \bar{\mathbf{y}}\mathbf{b} = \begin{bmatrix} 0 & 0 & 15 & -5 \end{bmatrix} \begin{bmatrix} 100 \\ 70 \\ 30 \\ 2 \end{bmatrix} = 440.$

Exercise 3.5. Consider the following LP:

$$\begin{aligned} \textbf{max} \quad & z = 3x_1 + 7x_2 + 5x_3 \\ \textbf{s.t.} \quad & x_1 + \ \ x_2 + x_3 \ \leq \ 50 \\ & 2x_1 + 3x_2 + x_3 \ \leq 100 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

After adding slack variables s_1 and s_2, the optimal tableau is as shown in the table below.

Basic	x_1	x_2	x_3	s_1	s_2	RHS
z	3	0	0	4	1	300
x_3	$1/2$	0	1	$3/2$	$-1/2$	25
x_2	$1/2$	1	0	$-1/2$	$1/2$	25

Using this optimal tableau, answer the following questions:

1. Find the shadow prices for the LP.
2. If the right-hand side is changed from 50 to 60, what would be the profit?

3.6 Duality and Sensitivity Analysis

From the dual theorem we demonstrate the following: Assuming that a set of basic variables BV is feasible, then BV is optimal if and only if the associated dual solution $\mathbf{C}_{BV}\mathbf{B}^{-1}$ is dual feasible.

$$\text{Unbounded} \implies \text{infeasible}, \quad \text{then} \quad \text{feasible} \implies \text{bounded}$$

This result can be used for an alternative way of doing the following types of sensitivity analysis:

1. Changing the objective function coefficient of a nonbasic variable.
2. Changing a column of a nonbasic variable.
3. Adding a new variable.

Since primal optimality and dual feasibility are equivalent, the above changes will leave the current basic optimal if and only if the current dual solution $\mathbf{C}_{BV}\mathbf{B}^{-1}$ remains dual feasible.

Example 3.9. Consider the following LP.

$$\begin{aligned}
\textbf{max} \quad & z = 60x_1 + 30x_2 + 20x_3 \\
\textbf{s.t.} \quad & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + \frac{3}{2}x_3 \leq 20 \\
& 2x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 \leq 8 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

The dual of the problem is:

$$\begin{aligned}
\text{min} \quad & w = 48y_1 + 20y_2 + 8y_3 \\
\text{s.t.} \quad & 8y_1 + 4y_2 + 2\,y_3 \geq 60 \\
& 6y_1 + 2y_2 + \frac{3}{2}y_3 \geq 30
\end{aligned}$$

$$\begin{aligned} y_1 + \frac{3}{2}y_2 + \frac{1}{2}y_3 \ &\geq 20 \\ y_1,\ y_2,\ y_3 &\geq 0 \end{aligned}$$

The optimal solution for the primal was $z = 280$, $\underbrace{s_1 = 24, x_3 = 8, x_1 = 2}_{\text{Basic}}$, $\underbrace{x_2 = s_2 = s_3 = 0}_{\text{NonBasic}}$. Also, the optimal dual solution (constraint shadow prices) are $y_1 = 0, y_2 = 10, y_3 = 10$.

1. Let c_2 be the coefficient of x_2 in the objective function. For what values of c_2 will the current basis remain optimal?

   Solution: If $y_1 = 0, y_2 = 10, y_3 = 10$ remains dual feasible, then the current basis and the values of all the variables are unchanged. Note that if the objective function coefficient for x_2 is changed, then the first and third dual constraints remain unchanged, but the second dual constraint is changed to $6y_1 + 2y_2 + \frac{3}{2}y_3 \geq c_2$. Thus, the current basis remains optimal if c_2 satisfies $6(0) + 2(10) + \frac{3}{2}(10) \geq c_2$, or $c_2 \leq 35$.

2. Suppose we change the elements of the column for x_2 from $\begin{bmatrix} 30 \\ \hline 6 \\ 2 \\ 3/2 \end{bmatrix}$ to $\begin{bmatrix} 43 \\ \hline 5 \\ 2 \\ 2 \end{bmatrix}$, does the current basis remain optimal?

   Solution: Changing the column for the nonbasic variable leaves the first and third dual constraints unchanged but changes the second to

   $$5y_1 + 2y_2 + 2y_3 \geq 43$$

   Because $y_1 = 0, y_2 = 10, y_3 = 10$ does not satisfy the new second dual constraint, dual feasibility is not maintained, and the current basis is no longer optimal.

3. Suppose we add new activity x_4 to the problem, and we add of the x_4 column $\begin{bmatrix} 15 \\ \hline 1 \\ 1 \\ 1 \end{bmatrix}$ to the problem. Does the current basis remain optimal?

   Solution: Introducing the new activity leaves the three dual constraints unchanged, but the new variable x_4 adds a new dual constraint. The new dual constraint will be $y_1 + y_2 + y_3 \geq 15$. Because $0 + 10 + 10 \geq 15$, the current basis remains optimal.

Exercise 3.6. Consider the following LP and its optimal tableau:

$$\begin{array}{ll} \textbf{max} & z = 5x_1 + x_2 + 2x_3 \\ \textbf{s.t.} & x_1 + x_2 + 1x_3 \le 6 \\ & 6x_1 \qquad + 1x_3 \le 8 \\ & \qquad x_2 + 1x_3 \le 2 \\ & x_1,\ x_2,\ x_3 \ge 0 \end{array}$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	RHS
z	0	$1/6$	0	0	$5/6$	$1/6$	9
s_1	0	$1/6$	0	1	$-1/6$	$-5/6$	3
x_1	1	$-1/6$	0	0	$1/6$	$-1/6$	1
x_3	0	1	1	0	0	1	2

1. Find the dual to this LP and its optimal solution.
2. Find the range of values of the objective function coefficient of x_2 for which the current basis remains optimal.
3. Find the range of values of the objective function coefficient of x_1 for which the current basis remains optimal.

3.7 The Dual–Simplex Method

In the *simplex algorithm* presented in Chapter 2 the problem starts at a (basic) feasible solution. Successive iterations continue to be feasible until the optimal is reached at the last iteration. The algorithm is sometimes referred to as the primal simplex method.

This section presents two additional algorithms: The *dual simplex* and the *generalized simplex*.

- In the *dual simplex*, the LP starts at a better than optimal infeasible (basic) solution. Successive iterations remain infeasible and (better than) optimal until feasibility is restored at the last iteration.
- The *generalized simplex* combines both the primal and dual simplex methods in one algorithm. It deals with problems that start both non-optimal and infeasible. In this algorithm, successive iterations are associated with basic feasible or infeasible (basic) solutions. At the final iteration, the solution becomes optimal and feasible (assuming that one exists).

Dual Simplex Algorithm: The crux of the dual simplex method is to start with a better than optimal and infeasible basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while moving the solution iterations toward feasibility.

- To start the LP optimal and infeasible, two requirements must be met:
  - The objective function must satisfy the optimality condition of the regular simplex method.
  - All the constraints must be of the type ($\leq$), regardless the type of problem either max or min. This condition requires converting any ($\geq$) to ($\leq$) simply by multiplying both sides of the inequality ($\geq$) by -1. If the LP includes ($=$) constraints, the equation can be replaced by two inequalities. For example, $x_1 + x_2 = 1$ is equivalent to $\begin{cases} x_1 + x_2 & \leq 1 \\ x_1 + x_2 & \geq 1 \end{cases}$ or $\begin{cases} x_1 + x_2 & \leq 1 \\ -x_1 - x_2 & \leq -1 \end{cases}$
- After converting all the constraints to ($\leq$), the starting solution is infeasible if at least one of the right-hand sides of the inequalities is strictly negative.
- **Dual feasibility condition**. The leaving variable, x_r is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.
- **Dual optimality condition**. Given that x_r is the leaving variable, let $\overline{c}_j$ be the reduced cost of nonbasic variable x_j and a_{rj} the constraint coefficient in the x_r-row and x_j-column of the tableau. The entering variable is the nonbasic variable with $a_{rj} < 0$ that corresponds to

$$\min_{\text{NonBasic } x_j} \left\{ \left| \frac{\overline{c}_j}{a_{rj}} \right| \ , \ a_{rj} < 0 \right\}$$

  Note that

  1. Ties are broken arbitrarily.
  2. If $a_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

Example 3.10. Use the dual−simplex algorithm for solving the following LP problem.

$$\begin{aligned} \textbf{min} \quad & w = 5x_1 + 6x_2 \\ \textbf{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 4x_1 - x_2 \geq 4 \\ & x_1, \ x_2 \geq 0 \end{aligned}$$

Solution: After converting all the constraints to ($\leq$), then adding slack variables s_1 and s_2 to the constraints, the LP in standard form is

$$\begin{aligned} \textbf{min} \quad & w - 5x_1 - 6x_2 = 0 \\ \textbf{s.t.} \quad & -\ x_1 - x_2 + s_1 \qquad = -2 \\ & -4x_1 + x_2 \qquad + s_2 \ = -4 \\ & x_1,\ x_2,\ s_1,\ s_2 \geq 0 \end{aligned}$$

The initial tableau and all following tableaus, using the dual-simplex algorithm, are shown below.

Iteration [0]	Basic	↓ x_1	x_2	s_1	s_2	RHS	**Optimal but not feasible**
	w	-5	-6	0	0	0	
	s_1	-1	-1	1	0	-2	
←	s_2	-4	1	0	1	-4	

Iteration [1]	Basic	x_1	x_2	s_1	↓ s_2	RHS	**Optimal but not feasible**
	w	0	$-29/4$	0	$-5/4$	5	
←	s_1	0	$-5/4$	1	$-1/4$	-1	
	x_1	1	$-1/4$	0	$-1/4$	1	

Iteration [2]	Basic	x_1	x_2	s_1	s_2	RHS	**Optimal and feasible**
	w	0	-1	-5	0	10	
	s_2	0	5	-4	1	4	
	x_1	1	1	-1	0	2	

Example 3.11. Use the dual−simplex algorithm for solving the following LP problem.

$$\begin{aligned} \textbf{max} \quad & z = -4x_1 - 2x_2 \\ \textbf{s.t.} \quad & x_1 + x_2 = 1 \\ & -\ 3x_1 + x_2 \geq 2 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Solution: After converting all the constraints to $(\leq)$, then adding slack variables s_1, s_2 and s_3 to the constraints, the LP in standard form is

$$\begin{aligned} \textbf{max} \quad & z + 4x_1 + 2x_2 = 0 \\ \textbf{s.t.} \quad & x_1 + x_2 + s_1 \qquad\qquad = \ \ 1 \\ & -\ x_1 - x_2 \qquad + s_2 \qquad = -1 \\ & 3x_1 - x_2 \qquad\qquad + s_3 \ = -2 \\ & x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0 \end{aligned}$$

The initial tableau and all following tableaus, using the dual-simplex algorithm, are shown below.

Iteration [0]

	Basic	x_1	$\downarrow$ x_2	s_1	s_2	s_3	RHS
	z	4	2	0	0	0	0
	s_1	1	1	1	0	0	1
	s_2	-1	-1	0	1	0	-1
$\leftarrow$	s_3	3	-1	0	0	1	-2

Optimal but not feasible

Iteration [1]

	Basic	x_1	x_2	s_1	s_2	s_3	RHS
	z	10	0	0	0	2	-4
$\leftarrow$	s_1	4	0	1	0	1	-1
	s_2	-4	0	0	1	-1	1
	x_2	-3	1	0	0	-1	2

Since s_1 leaves with no entering variable, then the solution is infeasible

Note 15. The dual simplex method is often used to find the new optimal solution to an LP after a constraint is added. When a constraint is added, one of the following three cases will occur:

1. The current optimal solution satisfies the new constraint.
2. The current optimal solution does not satisfy the new constraint, but the LP still has a feasible solution.
3. The additional constraint causes the LP to have no feasible solutions.

Example 3.12. Consider the following LP and its optimal tableau.

$$\begin{array}{ll} \textbf{max} & z = 6x_1 + x_2 \\ \textbf{s.t.} & \quad x_1 + \quad x_2 \quad \leq 5 \\ & 2 \;\; x_1 + \quad x_2 \quad \leq 6 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Basic	x_1	x_2	s_1	s_2	RHS
z	0	2	0	3	18
s_1	0	$1/2$	1	$-1/2$	2
x_1	1	$1/2$	0	$1/2$	3

Find the optimal solution to this LP if we add the constraint

1. $3x_1 + x_2 \leq 10$.

   Solution: After converting the constraint to (=) by adding s_3 to the constraint, we obtain:

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
s_3	$\boxed{3}$	1	0	0	1	10

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
s_3	0	$-1/2$	0	$-3/2$	1	1

The solution is optimal and feasible

2. $x_1 - 2x_2 \geq 6$.

   Solution: After converting the constraint to $(\leq)$ then adding s_3 to the constraint, we obtain:

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
s_3	$\boxed{-1}$	2	0	0	1	-6

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
s_3	0	$5/2$	0	$1/2$	1	-3

Since s_3 leaves with no entering variable, then the solution is infeasible

3. $8x_1 + x_2 \leq 12$.

   Solution: After converting the constraint to $(=)$ by adding s_3 to the constraint, we obtain:

Iteration [0]

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
s_3	$\boxed{8}$	1	0	0	1	12

↓

Iteration [0]

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	2	0	3	0	18
s_1	0	$1/2$	1	$-1/2$	0	2
x_1	1	$1/2$	0	$1/2$	0	3
← s_3	0	-3	0	-4	1	-12

The solution is optimal but infeasible

Iteration [1]

Basic	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	0	$1/3$	$2/3$	10
s_1	0	0	1	$-7/3$	$1/6$	0
x_1	1	0	0	$-1/6$	$1/6$	1
x_2	0	1	0	$4/3$	$-1/3$	4

The solution is optimal and feasible

Generalized Simplex Algorithm: The (primal) simplex algorithm starts feasible but not optimal. The dual simplex starts (better than) optimal but infeasible. What if an LP model starts both not optimal and infeasible? The following example illustrates what we call the generalized simplex algorithm for solving LP problems with this situation.

Example 3.13. Consider the following LP.

$$
\begin{aligned}
\textbf{max} \quad & z = x_1 - 3x_2 \\
\textbf{s.t.} \quad & x_1 - \ \ x_2 \le 2 \\
& x_1 + \ \ x_2 \ge 4 \\
& 2x_1 + 2x_2 \ge 3 \\
& x_1,\ x_2 \ge 0
\end{aligned}
$$

The model can be put in the following tableau form in which the starting basic solution (s_1, s_2, s_3) is both non-optimal (because x_1 has a negative reduced cost) and infeasible (because $s_2 = -4$, $s_3 = -3$).

Iteration [0]	Basic	↓ x_1	x_2	s_1	s_2	s_3	RHS	**The solution is not optimal and infeasible**
	z	-1	3	0	0	0	0	
	s_1	1	-1	1	0	0	2	
←	s_2	-1	-1	0	1	0	-4	
	s_3	-2	-2	0	0	1	-3	

Remove infeasibility first by applying a version of the dual simplex feasibility condition that selects s_2 as the leaving variable. To determine the entering variable, all we need is a nonbasic variable whose constraint coefficient in the s_2-row is strictly negative. The selection can be done without regard to optimality, because it is nonexistent at this point anyway. In the present example, x_1 and x_2 have negative coefficient in the s_2-row and x_1 is selected as the entering variable. The result is the following tableau:

Iteration [1]	Basic	x_1	↓ x_2	s_1	s_2	s_3	RHS	**The solution is not optimal and infeasible**
	z	0	4	0	-1	0	4	
←	s_1	0	-2	1	1	0	-2	
	x_1	1	1	0	-1	0	4	
	s_3	0	0	0	-2	1	5	

At this point, s_1 leaves the solution and x_2 have negative coefficient in the s_1-row and is selected as the entering variable. The result is the following tableau:

Iteration [2]	Basic	x_1	x_2	s_1	s_2	s_3	RHS	**The solution is optimal and feasible**
	z	0	0	2	1	0	0	
	x_2	0	1	$-1/2$	$-1/2$	0	1	
	x_1	1	0	$1/2$	$-1/2$	0	3	
	s_3	0	0	0	-2	1	5	

The solution in the preceding tableau is now feasible and fortunately optimal.

Exercise 3.7.

1. Solve the following LP by using the generalized simplex algorithm.

$$\begin{aligned} \textbf{max} \quad & z = 2x_1 - x_2 + x_3 \\ \textbf{s.t.} \quad & 2\,x_1 + 4x_2 - \ \ x_3 \ \geq 4 \\ & -x_1 + \ \ x_2 - \ \ x_3 \ \geq 2 \\ & \ \ x_1 + 2x_2 + 2x_3 \ \leq 8 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

2. Use the dual simplex method to solve the following LP:

$$\begin{aligned} \textbf{max} \quad & z = -2x_1 - x_3 \\ \textbf{s.t.} \quad & x_1 + \ \ x_2 - \ \ x_3 \ \geq 5 \\ & x_1 - 2x_2 + 4x_3 \ \geq 8 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

3. Solve the following LP in three different ways. Which method appears to be the most efficient computationally?

$$\begin{aligned} \textbf{min} \quad & w = 6x_1 + 7x_2 + 3x_3 + 5x_4 \\ \textbf{s.t.} \quad & 5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12 \\ & \qquad\quad\ x_2 - 5x_3 - 6x_4 \geq 10 \\ & 2x_1 + 5x_2 + \ \ x_3 + \ \ x_4 \geq 8 \\ & x_1,\ x_2,\ x_3,\ x_4 \geq 0 \end{aligned}$$

Answers

DON'T EVEN DARE PEEK AT THE SOLUTIONS TO AN EXERCISE UNTIL YOU'VE GENUINELY TRIED TO SOLVE THE EXERCISE !!

Chapter 1 Exercise 1.1

1. Let $x_1 =$ number of hours of Process 1, $x_2 =$ number of hours of Process 2. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{min} & w = 4x_1 + x_2 \\ \textbf{s.t.} & 3x_1 + x_2 \geq 10 \\ & x_1 + x_2 \geq 5 \\ & x_1 \geq 3 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

2. Let $x_1 =$ number of units of A, $x_2 =$ number of units of B. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{max} & z = 20x_1 + 50x_2 \\ \textbf{s.t.} & x_1 \geq 0.80\,(x_1 + x_2) \\ & x_1 \leq 100 \\ & 2x_1 + 4x_2 \leq 240 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

3. Let $x_1 =$ number of units of food F1 to be eaten, $x_2 =$ number of units of food F2 to be eaten. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{min} & w = 0.05x_1 + 0.03x_2 \\ \textbf{s.t.} & 2x_1 + x_2 \geq 400 \\ & x_1 + 2x_2 \geq 500 \\ & 4x_1 + 4x_2 \geq 1400 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

4. Let $x_1 =$ pounds of food A purchased, $x_2 =$ pounds of food B purchased. Then, the formulation of the problem is

$$\begin{array}{ll} \textbf{min} & w = 1.3x_1 + 0.8x_2 \\ \textbf{s.t.} & 5x_1 + 2x_2 \geq 60 \\ & 3x_1 + 2x_2 \geq 45 \\ & 4x_1 + x_2 \geq 30 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

5. Let x_1: number of computers of standard model, x_2: number of computers of portable model. Then the LP formulation is

$$\begin{array}{ll} \textbf{max} & z = 320x_1 + 220x_2 \\ \textbf{s.t.} & 400x_1 + 250x_2 \leq 20000 \\ & 40x_1 + 30x_2 \leq 2160 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

6. Let x_1 = number of acres of crop A, x_2 = number of acres of crop B, and x_3 = number of acres of crop C. Then, the LP formulation is

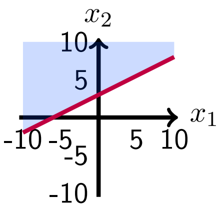
$$\begin{array}{ll} \textbf{max} & z = 100x_1 + 300x_2 + 200x_2 \\ \textbf{s.t.} & x_1 + x_2 + x_3 \leq 100 \\ & 40x_1 + 20x_2 + 30x_3 \leq 3200 \\ & x_1 + 2x_2 + x_3 \leq 160 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

7. Let x_1 be the number of chocolate cakes baked, and x_2 be the number of vanilla cakes baked. Then, the LP formulation is

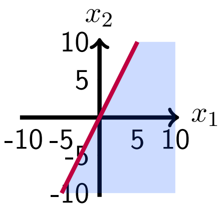
$$\begin{array}{rl} \textbf{max} & z = x_1 + 0.5x_2 \\ \textbf{s.t.} & 20x_1 + 40x_2 \leq 480 \\ & 4\ x_1 + \quad x_2 \leq \ \ 30 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Chapter 1 Exercise 1.2

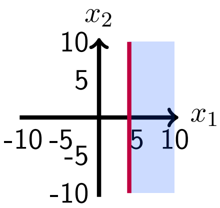
1.



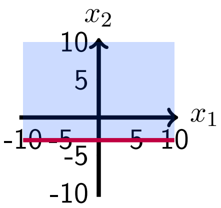
(A)



(B)



(C)



(D)

2. **Hyperplane:** (a), (e). **Half-Space:** (b). **Neither:** (c), (d), (f)

3. **Convex:** (A), (B), (F), (H)

4. (a) $\big(1+t, 3+t, 2-3t\big)\ ;\ 0 \leq t \leq 1$
   (b) $\big(1+t, 3+t, 2-3t\big) = \big(1.5, 3.5, 0.5\big)$ if $t = 0.5 \in \big[0, 1\big]$

Chapter 1 Exercise 1.3

1. (a) IV (b) II (c) I (d) III

2. (a) 16 (b) 55 (c) 84 (d) 90

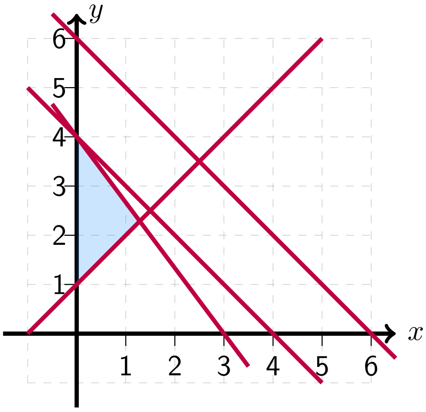
3. (a) 32 (b) 55 (c) 36 (d) 32

4. (a) $2a < b$ (b) $\frac{a}{3} < b < 2a$ (c) $b < \frac{a}{3}$ (d) $b = 2a$ (e) $b = \frac{a}{3}$

5. (a) south-east (down-right) (b) south-west (down-left) (c) north-west (up-left)

6. (a) north-west (up-left) (b) south-east (down-right)

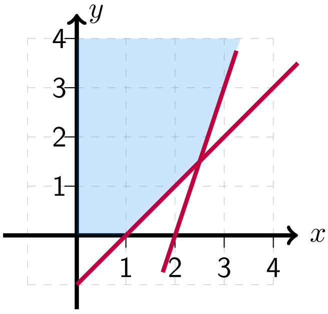
7. The constraints $x + y \leq 6$ and $x + y \leq 4$ are redundant.



8. The redundant constraint is $y \leq 3$

$$\begin{aligned} -x + y &\leq 1 \\ x + y &\leq 5 \\ x - 2y &\leq 2 \\ y &\leq 3 \\ x &\geq 1 \\ x, y &\geq 0 \end{aligned}$$

9.



10. *the solution is left to the student.*

11. $x_1 = 2$, $x_2 = 3$ by solving the two constraint-equations.

12. (a) $w = 12$, $x_1 = 3$, $x_2 = 0$
    (b) unbounded optimal solution
    (c) $w = 6$, $x_1 = 0$, $x_2 = 6$
    (d) $z = 4$, $x_1 = 0$, $x_2 = 6$
    (e) $z = 16$, $x_1 = 6$, $x_2 = 2$
    (f) $w = 14$, $x_1 = 3$, $x_2 = 2$

Chapter 2 Exercise 2.1

1. **max** $z = 2x_1 + 3x_2 + 5y_3 + 5y_4$
   s.t. $-x_1 - x_2 + y_3 + y_4 + s_1 = 5$
   $-6x_1 + 7x_2 - 9y_3 - 9y_4 + s_2 = 4$
   $x_1 + x_2 + 4y_3 + 4y_4 = 10$
   $x_1, x_2, y_3, y_4, s_1, s_2 \geq 0$

2. $22x_1 - 4x_2 \geq -7 \Rightarrow 22x_1 - 4x_2 - e1 = -7 \Rightarrow -22x_1 + 4x_2 + e_1 = 7$
   $22x_1 - 4x_2 \geq -7 \Rightarrow -22x_1 + 4x_2 \leq 7 \Rightarrow -22x_1 + 4x_2 + s_1 = 7$

3.

x	y_1	y_2
-6	0	6
10	10	0
0	0	6

Chapter 2 Exercise 2.2

1. (a) **max** $z = 2x_1 + 3x_2$
   s.t. $x_1 + 3x_2 + s_1 = 12$
   $3x_1 + 2x_2 + s_2 = 12$
   $x_1, x_2, s_1, s_2 \geq 0$

(b)

NBVs	BVs	values	feasible?	$z-$value
x_1, x_2	s_1, s_2	$(12, 12)$	yes	0
x_2, s_2	s_1, x_1	$(8, 4)$	yes	8
x_1, s_2	s_1, x_2	$(-6, 6)$	no	
s_1, x_2	s_2, x_1	$(-24, 12)$	no	
x_1, s_1	s_2, x_2	$(4, 4)$	yes	12
s_1, s_2	x_1, x_2	$(12/7, 24/7)$	yes	$96/7$

(c) from the table above, the optimum solution is $z = 96/7$, $x_1 = 12/7$, $x_2 = 24/7$.

(d) *the solution is left to the student*

(e) *the solution is left to the student*

2. (a) *the solution is left to the student*

(b)

BVs	values	feasible?	$w-$value
x_1, x_3	$(4, 0)$	yes	4
x_1, x_4	$(4, 0)$	yes	4
x_2, x_3	$(2, 0)$	yes	4
x_2, x_4	$(2, 0)$	yes	4
x_3, x_4	$(-4/7, 16/7)$	no	

3.

BVs	values	feasible?	$z-$value
s_1, e_2	$(3/2, -8)$	no	
s_1, x_1	$(-1, 4)$	no	
s_1, x_2	$(-13, 8)$	no	
e_2, x_1	$(-5, 3)$	no	
e_2, x_2	$(-13/2, 3/2)$	no	
x_1, x_2	$(13/3, -2/3)$	no	

4. (a)

BVs	values	feasible?	$z-$value
s_1, s_2	$(2, 4)$	yes	0
s_1, x_1	$(4, -2)$	yes	-2
s_1, x_2	$(-2, 4)$	no	
s_2, x_1	$(8, 2)$	yes	2
s_2, x_2	$(2, 2)$	yes	6
x_1, x_2	$(-2/3, 8/3)$	yes	$22/3$

(b) see the table above

(c) *the solution is left to the student*

Chapter 2 Exercise 2.3

1. (a) $z = 32/3$, $x_1 = 10/3$, $x_2 = 4/3$
   (b) $z = 25$, $x_1 = 15$, $x_2 = 5$, $x_3 = 0$
   (c) $z = 25$, $x_1 = 25$, $x_2 = 0$

   (d) $z = 12$, $x_1 = 4$, $x_2 = 4$, $x_3 = 4$

2.

BVs	values	feasible?	$z-$value
x_1	30	yes	150
x_2	10	yes	-60
x_3	6	yes	18
x_4	5	yes	-25
x_5	10	yes	120
s_1	30	yes	0

Chapter 2 Exercise 2.4

1. $w = -5$, $x_1 = 0$, $x_2 = 5$
2. $w = -2$, $x_1 = 0$, $x_2 = 2$
3. $w = -7.5$, $x_1 = 0$, $x_2 = 1.5$
4. $w = -9$, $x_1 = 3$, $x_2 = 0$
5. $w = -48$, $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, $x_4 = 8$

Chapter 2 Exercise 2.5

1. (a) $w = 1$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$
   (b) $w = 2$, $x_1 = 2$, $x_2 = 0$, $x_3 = 0$
   (c) $w = 4$, $x_1 = 2$, $x_2 = 0$
   (d) $z = 5$, $x_1 = 1$, $x_2 = 2$
2. $w = \nicefrac{214}{7}$, $x_1 = \nicefrac{66}{7}$, $x_2 = \nicefrac{8}{7}$, $x_3 = 0$, $x_4 = 0$
3. $z = 10$, $x_1 = 4$, $x_2 = 0$, $x_3 = 2$

Chapter 2 Exercise 2.6

1. (a) iterations 2 and 3 are degenerate, and degeneracy is removed in iteration 4.
   (b) *the solution is left to the student*
   (c) 3 iterations
2. $z = 10$ when: $x_1 = 0, x_2 = 0, x_3 = \nicefrac{10}{3}$
   $x_1 = 0, x_2 = 5, x_3 = 0$
   $x_1 = 1, x_2 = 4, x_3 = \nicefrac{1}{3}$

3. x_3 and s_1 can yield alternative optima, but because all their constraint coefficients are non-positive, non can yield an alternative basic solution.

Basic	x_1	x_2	x_3	s_1	s_2	RHS
z	0	0	0	0	1	20
x_1	1	0	-2	-1	0	15
s_2	0	1	-7	-2	1	10

4. the optimal solution is degenerate because s_3 is basic and equal 0, also it has alternative nonbasic solution because s_2 has zero coefficient in $z-$row and all its coefficients are ≤ 0.

Basic	x_1	x_2	x_3	s_1	s_2	s_3	RHS
z	0	5	0	3	0	0	15
x_3	0	1	1	1	-1	0	3
x_1	1	2	0	1	0	0	5
s_3	0	-6	0	-2	-5	1	0

5. (a) solution space unbounded in the direction of x_3
   (b) objective value is unbounded because each unit increase in x_3, increases z by 1.

6. because $a_2 = 0$ in the optimal tableau, the problem has feasible optimal solution: $x_1 = 0$, $x_2 = 4$, $z = 8$.

Basic	x_1	x_2	x_3	s_1	e_2	a_2	RHS
z	$5M-1$	0	$2M-1$	M	$4M+2$	0	8
x_2	2	1	1	0	1	0	4
a_2	-5	0	-2	-1	-4	1	0

7. (a) BVs : $(x_8, x_3, x_1) = (12, 6, 0)$; $z = 620$

   NBVs : $(x_2, x_4, x_5, x_6, x_7) = (0, 0, 0, 0, 0)$

   (b) x_2 enters: $x_2 = \min\left(\frac{12}{3}, \frac{6}{1}, -\right) = 4$. so x_8 leaves with $\Delta z = 5 \times 4 = 20$.

   x_5 enters: $x_5 = \min\left(-, \frac{6}{1}, \frac{0}{6}\right) = 0$. so x_1 leaves with $\Delta z = 1 \times 0 = 0$.

   x_6 enters: $x_2 = \min\left(-, -, -\right) = 4$. so no leaving variable leaves and x_6 can be increased to ∞ with $\Delta z = \infty$.

   (c) *the solution is left to the student*

   (d) x_5 and x_7

8. (a) $c < 0$, $b \geq 0$
   (b) $c = 0$, $b \geq 0$, $a_2 > 0$ and/or $a_3 > 0$. If only $a_3 > 0$ then $b > 0$
   (c) $c > 0$, $a_2 \leq 0$, $a_3 \leq 0$

9. (a) $b \geq 0$ is necessary.
    - If $c_1 = 0$ and $c_2 \geq 0$ we can pivot in x_1 to obtain an alternative optimum.
    - If $c1 \geq 0$, $c_2 \geq 0$ and $a_2 > 0$ we can pivot in x_5 and obtain an alternative optimum.
    - If $c_2 = 0$, $a_1 > 0$ and $c_1 \geq 0$ we can pivot in x_2 and obtain an alternative optimum.

   (b) $b < 0$

   (c) $b = 0$

   (d) $b \geq 0$ makes the solution feasible. If $c_2 < 0$ and $a_1 \leq 0$ we can make x_2 as large as desired and obtained an unbounded solution.

   (e) $b \geq 0$ makes the current basic solution feasible. For x_6 to replace x_1 we need $c_1 < 0$ (this ensures that increasing x_1 will increase z) and we need Row 3 to win the ratio test for x_1. This requires $3/a_3 \leq b/4$.

10. *the solution is left to the student*

Chapter 3 Exercise 3.1

1.

Basic	x_1	x_2	s_1	s_2	RHS
z	0	0	4	5	28
x_1	1	0	1	1	6
x_2	0	1	1	2	10

2.

Basic	x_1	x_2	s_1	s_2	RHS
z	2	0	0	1	2
x_2	1	1	0	1	2
s_1	1	0	1	-1	2

3. *the solution is left to the student*

4.

Basic	x_1	x_2	e_1	e_2	RHS
z	0	0	-5	$-15/2$	3800
x_1	1	0	$-3/20$	$1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$7/5$

Chapter 3 Exercise 3.2

1. $\alpha \geq -1$

2. (a) $\overline{z} = 26$

   (b) $\Delta \geq -2$

   (c) $\Delta \leq 5$

   (d) NO

Chapter 3 Exercise 3.3

1. **min** $w = y_1 + 3y_2 + 4y_3$
   s.t. $-y_1 + y_2 + y_3 \geq 2$
   $y_1 + y_2 - 2y_3 \geq 1$
   $y_1, y_2, y_3 \geq 0$

2. **max** $z = 4x_1 + x_2 + 3x_3$
   s.t. $2x_1 + x_2 + x_3 \leq 1$
   $x_1 + x_2 + x_3 \leq -1$
   $x_1, x_2, x_3 \geq 0$

3. **min** $w = 5y_1 + 7y_2 + 6y_3 + 4y_4$
   s.t. $y_1 + 2y_2 + y_4 \geq 4$
   $y_1 + y_2 + 2y_3 = -1$
   $y_3 + y_4 = 2$
   $y_1, y_2 \geq 0$, $y_3 \leq 0$, x_4 urs

4. **max** $z = 6x_1 + 8x_2$
   s.t. $x_1 + x_2 \leq 4$
   $2x_1 - x_2 \leq 2$
   $2x_2 = -1$
   $x_1 \leq$, x_2 urs

Chapter 3 Exercise 3.4

1. $w = 250/3$.

2. (a) *the solution is left to the student*
   (b) $\overline{\mathbf{C}}_{NBV} = \mathbf{C}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{C}_{NBV} = \begin{bmatrix}-2 & 7/2\end{bmatrix}$
   (c)
   - $\mathbf{y} = \mathbf{C}_{BV}\mathbf{B}^{-1} = \begin{bmatrix}4 & 0\end{bmatrix} = \begin{bmatrix}y_1 & y_2\end{bmatrix}$
   - Using the general formula: $y_1 = 0 + 4 = 4$
     $y_2 = 3 + (-3) = 0$

3. $\bar{z} = \underbrace{\mathbf{C}_{BV}\mathbf{B}^{-1}}_{\bar{\mathbf{y}}}\mathbf{b} = \begin{bmatrix}0.4 & 1.4\end{bmatrix}\begin{bmatrix}0.5 \\ 0.5\end{bmatrix} = 0.9$

4. since $\bar{\mathbf{y}} = \begin{bmatrix}0 & 1\end{bmatrix}$ then $\bar{\mathbf{w}} = 10 \neq 20/3$

5. *the solution is left to the student*

Chapter 3 Exercise 3.5

1. $y_1 = 4$, $y_2 = 1$

2. $z_{\text{new}} = z_{\text{old}} + 10 \times 4 = 340$

Chapter 3 Exercise 3.6

1. *the solution is left to the student*

2. $c_2 \leq 1/6$

3. $c_1 \leq 5/6$

Chapter 3 Exercise 3.7

1. $x_1 = {}^4\!/_3$, $x_2 = {}^{10}\!/_3$, $x_3 = 0$, $z = {}^2\!/_3$
2. $x_1 = 0$, $x_2 = 14$, $x_3 = 9$, $z = -9$
3. $x_1 = 0$, $x_2 = 10$, $x_3 = 0$, $x_4 = 0$, $w = 70$