"Applications of Laplace Transform to Differential Equations"

1. Ordinary differential equations with Constant Coefficients:

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. For example, suppose we wish to solve the 2nd order linear D.E $\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta y = f(t)$ or $Y'' + \alpha Y' + \beta Y = f(t)$ ...(1)

where α & β are Constants, subject to the initial or boundary conditions:

$Y(0) = A$, $Y'(0) = B$ ...(2) where A & B are given Constants.

On taking the $\mathcal{L}$ transform of both sides of Eq.(1) and using Eq.(2), we obtain an algebric equation for determination of $\mathcal{L}\{y(t)\} = y(s)$. The required solution is then obtained by finding the $\mathcal{L}^{-1}$ transform of $y(s)$. This method is easily extended to higher order differential equations.

Ex:- Solve $Y'' + Y = t$, $Y(0) = 1$, $y'(0) = -2$

$$\mathcal{L}\{Y'' + Y = t\} = s^2 y - s y(0) - y'(0) + y = \frac{1}{s^2}$$

$$s^2 y - s + 2 + y = \frac{1}{s^2} \Rightarrow (s^2+1) y = \frac{1}{s^2} + s - 2$$

$$\therefore y(s) = \frac{1}{s^2(s^2+1)} + \frac{s-2}{(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$\therefore y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1} \quad \therefore y(t) = t + \cos t - 3\sin t$$

Check: y, y', y'', then $Y'' + Y = t$.

(problems)

1. Solve: $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$ Hint $\mathcal{L}\{y''\} = s^2 y - s y(0) - y'(0)$, $\mathcal{L}\{y'\} = s y - y(0)$.

Ans: $y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}$.

check your answer.

2. Solve $Y''' - 3Y'' + 3Y' - Y = t^2 e^{t}$, $Y'(0) = 0$, $Y''(0) = -2$, $Y(0) = 1$

Hint: $\mathcal{L}\{y'''\} = s^3 y - s^2 y(0) - s Y'(0) - Y''(0)\}$, Hint: $\mathcal{L}\, t^2 e^{t} = \frac{2}{(s-1)^3}$

Ans: $y(t) = e^{t} - te^{t} - \frac{t^2 e^{t}}{2} + \frac{t^5 e^{t}}{60}$.

2. Odinary Differential Equations with Variable Coefficients:

The Laplace transform can also be used in solving some ordinary D.E in which the coefficients are variable. A particular D.E where the method proves useful is one in which the terms have the form $t^m y^{(n)}(t)$ the laplace transform of which is $(-1)^m \frac{d^m}{ds^m} \mathcal{L}\{y^{(n)}(t)\}$.

Example:- Solve $ty'' + y' + 4ty = 0$, $y(0) = 3$, $y'(0) = 0$

we have $\mathcal{L}\{ty''\} = -\frac{d}{ds}\{s^2 y - sy(0) - y'(0)\}$

$\mathcal{L}\{y'\} = sy - y(0)$, $\mathcal{L}\{4ty\} = -4\frac{dy}{ds}$

then: $-\frac{d}{ds}(s^2 y - sy(0) - y'(0)) + (sy - y(0)) - 4\, dy/ds = 0$

rearrange to obtain: $(s^2+4)\frac{dy}{ds} + sy = 0$

then $\frac{dy}{y} + \frac{s\,ds}{s^2+4} = 0$

integrate to get: $\ln y + \frac{1}{2}\ln(s^2+4) = c$, or $y = \frac{c}{\sqrt{s^2+4}}$

Inverting we find $Y = c\, J_0(2t)$ J_0 is the Bessel function of order zero

$Y(0) = c J_0(0) = c = 3 \quad \therefore Y = 3 J_0(2t)$.

Note: $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \cdots$ $\quad \mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

3. Simultaneous Ordinary Differential Equations:

Laplace transform can be used to solve two or more simultaneous ordinary D.E Ex:- solve

$$x' = 2x - 3y$$
$$y' = y - 2x$$

subject to $x(0) = 8$, $y(0) = 3$

Taking $\mathcal{L}$ transform, we have

$sx - 8 = 2x - 3y \Rightarrow (s-2)x + 3y = 8$

$sy - 3 = y - 2x \Rightarrow 2x + (s-1)y = 3$

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 3 & (s-1) \end{vmatrix}}{\begin{vmatrix} (s-2) & 3 \\ 2 & (s-1) \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{(s+1)} + \frac{3}{s-4} \quad \therefore x(t) = 5e^{-t} + 3e^{4t}$$

$$y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} (s-2) & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{(s+1)} - \frac{2}{(s-4)} \quad \therefore y(t) = 5e^{-t} - 2e^{4t}$$

problem: Solve

$$x'' + y' + 3x = 15e^{-t}$$
$$y'' - 4x' + 3y = 15\sin 2t$$

$x(0) = 35$, $x'(0) = -48$, $y(0) = 27$, $y'(0) = -55$

Ans: $x(t) = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$

$y(t) = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$

4. Applications:

(a) To Mechanics Systems:

A particular P of mass (2) grams moves on the X-axis and is attracted toward origin O with a force of 8X. if it is initially at rest at $x=10$, find its position at any subsequent time assuming (a) no other forces act, (b) a damping force numerically equal to 8 times the instantanous velocity acts.

Solution: (a) Mass . Acceleration = Net force

$$2 . X'' = -8x \Rightarrow X''+4X=0 \quad X(0)=10,\ X'(0)=0$$

then $s^2x - 10s + 4x = 0$ or $x = \frac{10s}{s^2+4}$ $\therefore X(t) = 10\cos 2t$

(b) Mass . Acceleration = Net force

$$2X'' = -8X - 8X' \Rightarrow X''+4X'+4X=0 \quad X(0)=10,\ X'(0)=0$$

then, $s^2x - 10s + 4(sx-10) + 4x = 0$ $\therefore x = \frac{10s+40}{s^2+4s+4}$

$$\therefore X(t) = \mathcal{L}^{-1}\{x\} = \mathcal{L}^{-1}\left\{\frac{10s+40}{(s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{10(s+2)+20}{(s+2)^2}\right\}$$

$$X(t) = 10\mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} + 20\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$X(t) = 10e^{-2t} + 20te^{-2t} = 10e^{-2t}(1+2t)$$

(b) Electrical Circuits:-

An inductor of 2 Henry, a resistor of 16 Ω and a capacitor of 0.02 F are connected in series with an e.m.f of E volts. At $t=0$ the charge on the capacitor and current in the circuit are zero. Find the charge and current at any time $t>0$ if (a) $E=300$ V, b $E = 100\sin 3t$ V.

Solution: let Q and I be the instantanous charge & current respectively at time t.

then by Kirchhoff's laws $2\frac{dI}{dt} + 16I + \frac{Q}{0.02} = E$, since $I = dQ/dt$

then $2Q'' + 16Q' + 50Q = E$ with $Q(0)=0$, $I(0) = Q'(0) = 0$

(a) $E = 300$ V then $\{s^2q - sq(0) - q'(0)\} + 8\{sq - q(0)\} + 25q = \frac{150}{s}$ (step function)

then $q(s) = \frac{6}{s} - \frac{6(s+4)}{(s+4)^2+9} - \frac{24}{(s+4)^2+9} \Rightarrow Q(t) = 6 - 6e^{-4t}\cos 3t - 8e^{-4t}\sin 3t$

$I(t) = \frac{dQ}{dt} = 50e^{-4t}\sin 3t$.

(b) for $E = 100\sin 3t$

Solve for $Q(t) = \frac{25}{52}(2\sin 3t - 3\cos 3t) + \frac{25}{52}e^{-4t}(3\cos 3t + 2\sin 3t)$. & $I(t) = \frac{dQ}{dt}$.

Appendix A

TABLE OF GENERAL PROPERTIES OF LAPLACE TRANSFORMS

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

	$f(s)$	$F(t)$
1.	$a f_1(s) + b f_2(s)$	$a F_1(t) + b F_2(t)$
2.	$f(s/a)$	$a F(at)$
3.	$f(s-a)$	$e^{at} F(t)$
4.	$e^{-as} f(s)$	$\mathcal{U}(t-a) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$
5.	$s f(s) - F(0)$	$F'(t)$
6.	$s^2 f(s) - s F(0) - F'(0)$	$F''(t)$
7.	$s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - F^{(n-1)}(0)$	$F^{(n)}(t)$
8.	$f'(s)$	$-t F(t)$
9.	$f''(s)$	$t^2 F(t)$
10.	$f^{(n)}(s)$	$(-1)^n t^n F(t)$
11.	$\frac{f(s)}{s}$	$\int_0^t F(u)\,du$
12.	$\frac{f(s)}{s^n}$	$\int_0^t \cdots \int_0^t F(u)\,du^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} F(u)\,du$
13.	$f(s)\,g(s)$	$\int_0^t F(u)\,G(t-u)\,du$

	$f(s)$	$F(t)$
14.	$\int_s^\infty f(u)\,du$	$\frac{F(t)}{t}$
15.	$\frac{1}{1-e^{-sT}}\int_0^T e^{-su}F(u)\,du$	$F(t) = F(t+T)$
16.	$\frac{f(\sqrt{s})}{s}$	$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}F(u)\,du$
17.	$\frac{1}{s}f(1/s)$	$\int_0^\infty J_0(2\sqrt{ut})\,F(u)\,du$
18.	$\frac{1}{s^{n+1}}f(1/s)$	$t^{n/2}\int_0^\infty u^{-n/2}J_n(2\sqrt{ut})\,F(u)\,du$
19.	$\frac{f(s+1/s)}{s^2+1}$	$\int_0^t J_0(2\sqrt{u(t-u)})\,F(u)\,du$
20.	$\frac{1}{2\sqrt{\pi}}\int_0^\infty u^{-3/2}e^{-s^2/4u}f(u)\,du$	$F(t^2)$
21.	$\frac{f(\ln s)}{s\ln s}$	$\int_0^\infty \frac{t^u F(u)}{\Gamma(u+1)}\,du$
22.	$\frac{P(s)}{Q(s)}$ $P(s)$ = polynomial of degree less than n, $Q(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct.	$\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$

Appendix B

TABLE OF SPECIAL LAPLACE TRANSFORMS

	$f(s)$	$F(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^n}$ $n = 1, 2, 3, \ldots$	$\frac{t^{n-1}}{(n-1)!}$, $0! = 1$
4.	$\frac{1}{s^n}$ $n > 0$	$\frac{t^{n-1}}{\Gamma(n)}$
5.	$\frac{1}{s-a}$	e^{at}
6.	$\frac{1}{(s-a)^n}$ $n = 1, 2, 3, \ldots$	$\frac{t^{n-1}e^{at}}{(n-1)!}$, $0! = 1$
7.	$\frac{1}{(s-a)^n}$ $n > 0$	$\frac{t^{n-1}e^{at}}{\Gamma(n)}$
8.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
9.	$\frac{s}{s^2+a^2}$	$\cos at$
10.	$\frac{1}{(s-b)^2+a^2}$	$\frac{e^{bt}\sin at}{a}$
11.	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
12.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
13.	$\frac{s}{s^2-a^2}$	$\cosh at$
14.	$\frac{1}{(s-b)^2-a^2}$	$\frac{e^{bt}\sinh at}{a}$

	$f(s)$	$F(t)$
15.	$\dfrac{s-b}{(s-b)^2-a^2}$	$e^{bt}\cosh at$
16.	$\dfrac{1}{(s-a)(s-b)}$ $\quad a \neq b$	$\dfrac{e^{bt}-e^{at}}{b-a}$
17.	$\dfrac{s}{(s-a)(s-b)}$ $\quad a \neq b$	$\dfrac{be^{bt}-ae^{at}}{b-a}$
18.	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{\sin at - at\cos at}{2a^3}$
19.	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t\sin at}{2a}$
20.	$\dfrac{s^2}{(s^2+a^2)^2}$	$\dfrac{\sin at + at\cos at}{2a}$
21.	$\dfrac{s^3}{(s^2+a^2)^2}$	$\cos at - \frac{1}{2}at\sin at$
22.	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
23.	$\dfrac{1}{(s^2-a^2)^2}$	$\dfrac{at\cosh at - \sinh at}{2a^3}$
24.	$\dfrac{s}{(s^2-a^2)^2}$	$\dfrac{t\sinh at}{2a}$
25.	$\dfrac{s^2}{(s^2-a^2)^2}$	$\dfrac{\sinh at + at\cosh at}{2a}$
26.	$\dfrac{s^3}{(s^2-a^2)^2}$	$\cosh at + \frac{1}{2}at\sinh at$
27.	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$	$t\cosh at$
28.	$\dfrac{1}{(s^2+a^2)^3}$	$\dfrac{(3-a^2t^2)\sin at - 3at\cos at}{8a^5}$
29.	$\dfrac{s}{(s^2+a^2)^3}$	$\dfrac{t\sin at - at^2\cos at}{8a^3}$
30.	$\dfrac{s^2}{(s^2+a^2)^3}$	$\dfrac{(1+a^2t^2)\sin at - at\cos at}{8a^3}$
31.	$\dfrac{s^3}{(s^2+a^2)^3}$	$\dfrac{3t\sin at + at^2\cos at}{8a}$

	$f(s)$	$F(t)$
32.	$\dfrac{s^4}{(s^2+a^2)^3}$	$\dfrac{(3-a^2t^2)\sin at + 5at\cos at}{8a}$
33.	$\dfrac{s^5}{(s^2+a^2)^3}$	$\dfrac{(8-a^2t^2)\cos at - 7at\sin at}{8}$
34.	$\dfrac{3s^2-a^2}{(s^2+a^2)^3}$	$\dfrac{t^2\sin at}{2a}$
35.	$\dfrac{s^3-3a^2s}{(s^2+a^2)^3}$	$\frac{1}{2}t^2\cos at$
36.	$\dfrac{s^4-6a^2s^2+a^4}{(s^2+a^2)^4}$	$\frac{1}{6}t^3\cos at$
37.	$\dfrac{s^3-a^2s}{(s^2+a^2)^4}$	$\dfrac{t^3\sin at}{24a}$
38.	$\dfrac{1}{(s^2-a^2)^3}$	$\dfrac{(3+a^2t^2)\sinh at - 3at\cosh at}{8a^5}$
39.	$\dfrac{s}{(s^2-a^2)^3}$	$\dfrac{at^2\cosh at - t\sinh at}{8a^3}$
40.	$\dfrac{s^2}{(s^2-a^2)^3}$	$\dfrac{at\cosh at + (a^2t^2-1)\sinh at}{8a^3}$
41.	$\dfrac{s^3}{(s^2-a^2)^3}$	$\dfrac{3t\sinh at + at^2\cosh at}{8a}$
42.	$\dfrac{s^4}{(s^2-a^2)^3}$	$\dfrac{(3+a^2t^2)\sinh at + 5at\cosh at}{8a}$
43.	$\dfrac{s^5}{(s^2-a^2)^3}$	$\dfrac{(8+a^2t^2)\cosh at + 7at\sinh at}{8}$
44.	$\dfrac{3s^2+a^2}{(s^2-a^2)^3}$	$\dfrac{t^2\sinh at}{2a}$
45.	$\dfrac{s^3+3a^2s}{(s^2-a^2)^3}$	$\frac{1}{2}t^2\cosh at$
46.	$\dfrac{s^4+6a^2s^2+a^4}{(s^2-a^2)^4}$	$\frac{1}{6}t^3\cosh at$
47.	$\dfrac{s^3+a^2s}{(s^2-a^2)^4}$	$\dfrac{t^3\sinh at}{24a}$
48.	$\dfrac{1}{s^3+a^3}$	$\dfrac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}at}{2} - \cos\dfrac{\sqrt{3}at}{2} + e^{-3at/2}\right\}$

	$f(s)$	$F(t)$
49.	$\dfrac{s}{s^3+a^3}$	$\dfrac{e^{at/2}}{3a}\left\{\cos\dfrac{\sqrt{3}\,at}{2}+\sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2}-e^{-3at/2}\right\}$
50.	$\dfrac{s^2}{s^3+a^3}$	$\dfrac{1}{3}\left(e^{-at}+2e^{at/2}\cos\dfrac{\sqrt{3}\,at}{2}\right)$
51.	$\dfrac{1}{s^3-a^3}$	$\dfrac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\cos\dfrac{\sqrt{3}\,at}{2}-\sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2}\right\}$
52.	$\dfrac{s}{s^3-a^3}$	$\dfrac{e^{-at/2}}{3a}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2}-\cos\dfrac{\sqrt{3}\,at}{2}+e^{3at/2}\right\}$
53.	$\dfrac{s^2}{s^3-a^3}$	$\dfrac{1}{3}\left(e^{at}+2e^{-at/2}\cos\dfrac{\sqrt{3}\,at}{2}\right)$
54.	$\dfrac{1}{s^4+4a^4}$	$\dfrac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$
55.	$\dfrac{s}{s^4+4a^4}$	$\dfrac{\sin at\sinh at}{2a^2}$
56.	$\dfrac{s^2}{s^4+4a^4}$	$\dfrac{1}{2a}(\sin at\cosh at+\cos at\sinh at)$
57.	$\dfrac{s^3}{s^4+4a^4}$	$\cos at\cosh at$
58.	$\dfrac{1}{s^4-a^4}$	$\dfrac{1}{2a^3}(\sinh at-\sin at)$
59.	$\dfrac{s}{s^4-a^4}$	$\dfrac{1}{2a^2}(\cosh at-\cos at)$
60.	$\dfrac{s^2}{s^4-a^4}$	$\dfrac{1}{2a}(\sinh at+\sin at)$
61.	$\dfrac{s^3}{s^4-a^4}$	$\frac{1}{2}(\cosh at+\cos at)$
62.	$\dfrac{1}{\sqrt{s+a}+\sqrt{s+b}}$	$\dfrac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$
63.	$\dfrac{1}{s\sqrt{s+a}}$	$\dfrac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
64.	$\dfrac{1}{\sqrt{s}\,(s-a)}$	$\dfrac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
65.	$\dfrac{1}{\sqrt{s-a}+b}$	$e^{at}\left\{\dfrac{1}{\sqrt{\pi t}}-b\,e^{b^2t}\operatorname{erfc}(b\sqrt{t})\right\}$