"State-Space Representation"

* Review to some elementry Matricies notation and operations:-

- Matrix : is a set of elements arranged in Raws and Columns.

Ex:- $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & \cdots & \cdots & & a_{mn} \end{pmatrix}$ which is $(m \times n)$ matrix where m = number of Raws, n = number of Columns.

with a_{ij} elements, i = no. of raw, j = no. of column.

- Real Matrix : with elements are all real.
- Imajinary Matrix: not all elements are real.
- Square Matrix: have the same numbers of Rows & columns.
- $(a_1 \; a_2 \; a_3)$ is called raw vector of (1×3) dimenssion.
- $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is called Column vector of (3×1) dimenssion.
- Matrix Addition (subtraction) : Matricies should have the same dimenssion.

Ex: $A = \underset{(3\times3)}{\begin{pmatrix} 3 & -1 & 2 \\ 5 & 1 & 0 \\ -2 & 1 & -3 \end{pmatrix}}$, $B = \underset{(3\times3)}{\begin{pmatrix} 2 & -2 & 1 \\ 3 & 1 & -4 \\ 2 & -1 & 1 \end{pmatrix}}$ then $A+B = \underset{(3\times3)}{\begin{pmatrix} 5 & -3 & 3 \\ 8 & 2 & -4 \\ 0 & 0 & -2 \end{pmatrix}}$

- Multiplication of Matricies: Multiplication exists if the inner dimenssions are equal, then the resultant matrix is of the outer dimenssions.

Ex: $(3\times2)(2\times1)$ Ans. (3×1) , $(2\times3)(2\times3)$ Ans. Multiplication is not valid.

- Transpose of a Matrix : Interchange Raws & Columns.

Ex: if $A = \underset{(3\times2)}{\begin{pmatrix} 2 & 3 \\ -1 & 2 \\ 3 & 1 \end{pmatrix}}$ then $A^T = \underset{(2\times3)}{\begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \end{pmatrix}}$

- Symmetrix Matrix if $A = A^T$, skew matrix if $A = -A^T$
- Hermitian Matrix if $A = \bar{A}^T$ skew Herimitian if $A = -\bar{A}^T$
- Complex Conjugate of a Matrix: if all elements of (A) are replaced by it's Complex conjugate then (A') is Complex conjugate of A.
- Unit Matrix (or Identity matrix) : if principal diagonal elements are all (1's) and other elements are zeroes Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Null Matrix : if all elements are zeroes.

- Determinant of a Matrix: for a square Matrix there is a determinant
Ex: $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, then $|A| = (1\times3 - 2\times4) = -5$ then A is called non-singular matix and invertible (i.e, A^{-1} inverse of a matrix is valid).
Ex: $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$, then $|A| = 0$ and A is called singular matrix and non-invertible.

- Inverse of a Matrix: if A is square and non-singular then (A^{-1}) inverse of a matrix is available with $A^{-1} = \frac{(\text{adjoint coefficient } a_{ij})^T}{\det. \text{ of } A}$
Ex:- $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ then $A^{-1} = \frac{\begin{pmatrix} 3 & -4 \\ -2 & 1 \end{pmatrix}^T}{-5} = \frac{\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}}{-5}$
then $A \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Scalar multiplication and division: if a matix A is multiplied or divided by scalar value all its elements are multiplied (divided) by this scalar. Ex: $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ then $2A = \begin{pmatrix} 2 & 4 \\ 8 & 6 \end{pmatrix}$.
Ex: $A = \frac{\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}}{-5}$ then $A = \begin{pmatrix} \frac{3}{-5} & \frac{-2}{-5} \\ \frac{-4}{-5} & \frac{1}{-5} \end{pmatrix}$.

- Eigen value are the roots of $|\lambda I - A| = 0$ equation.

* Why State-Space Representation:- "للإطلاع"

As a different mathematical Representation compared with that of classical mathematical Representation $\left(\frac{o/p}{i/p}\right)$ "Transfer Function Model" There are many reasons to use S.S. Representation. The following is a Comparison between the two approaches from the Control theories point of views.

S.S. Mathematical Representation	Classical (T.F) Math. Representation
1. Is easily applicable for Multi-Input-Multi-output (MIMO) and Single-Input/Single-output (SISO) Systems which may be linear or Non-linear, time invariant (TIV) or (TV).	1. Is easily applicable to linear time-invariant SISO systems.
2. Is essentially a time-domain approach.	2. Is a Complex Frequency-domain approach.
3. Helps to design optimal Control Systems.	3. Some approaches are based on trial and error procedures that will not yield to optimal sys. design.
4. Design will be carried out for a class of i/ps.	4. Design will be carried out for a specific i/p function.

* S.S. Mathematical Representation basic definitions:

1. State : is the smallest variable one can deal with. Ex: 2^{nd} order has 2-state.
2. State-space: is the n-dimenssional space whose coordinate axes consist of the $x_1, x_2, ---x_n$ axis , and any state can be represented by a point in the state-space.
3. State-space Equations:

$$\dot{\underline{x}} = A\underline{x} + Bu \quad \text{(state Equation)}$$

$$y = C\underline{x} + Du \quad \text{(output Equation)}$$

where; $\dot{\underline{x}}$ is the state-vector of $(n \times 1)$ elements , where n is the system order.
A is called the dynamic matrix of $(n \times n)$ dimenssion,
B is called the input matrix of $(n \times 1)$ dimenssion.
u is the input for SISO is of (1×1) dimenssion.
y is the output for SISO is of (1×1) dimenssion.
C is the output matrix of $(1 \times n)$ dimenssion.
D is the direct matrix "exists for proper T.F only" with (1×1) dimension.

* *Solution of state Equation* $\dot{\underline{x}} = A\underline{x} + Bu$

(a) *Solution of Homogenous (no-input) state Equation* $\dot{\underline{x}} = A\underline{x}$

$\dot{\underline{x}} = A\underline{x} \Rightarrow \quad SIX(s) - \underline{x}(0) = A\underline{x}(s) \Rightarrow (SI-A)\underline{x}(s) = x(0)$. $\quad x(t) = e^{At}\underline{x}(0)$

then $\underline{x}(s) = (SI-A)^{-1}\underline{x}(0) \Rightarrow \therefore \underline{x}(t) = \mathcal{L}^{-1}(SI-A)^{-1}\underline{x}(0)$ then $y(t) = C\underline{x}(t) + Du$

Note:- $\Phi(t) = e^{At} = \mathcal{L}^{-1}(SI-A)^{-1}$ = state-transition matrix.

(b) *Solution of Non-homogenous state Equation* $\dot{\underline{x}} = A\underline{x} + Bu$

$\dot{\underline{x}} = A\underline{x} + Bu \Rightarrow \quad S\underline{x}(s) - \underline{x}(0) = A\underline{x}(s) + Bu(s) \Rightarrow (SI-A)x(s) = x(0) + Bu(s) \Rightarrow$

$x(s) = (SI-A)^{-1}\underline{x}(0) + (SI-A)^{-1}B\underline{u}(s) \Rightarrow \quad x(t) = \mathcal{L}^{-1}(SI-A)^{-1}\underline{x}(0) + \underbrace{\mathcal{L}^{-1}(SI-A)^{-1}B\underline{u}(s)}_{\text{Convolution Integral}}$

then $\boxed{x(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)} B\underline{u}(\tau)\, d\tau}$, then $y(t) = C\underline{x}(t) + Du$.

* properties of state-transition matrix : $\Phi(t) = e^{At}$ = exponential matrix $= \mathcal{L}^{-1}(sI-A)^{-1}$

$\Phi(0) = I$, $\Phi^{-1}(t) = \Phi(-t)$, $\Phi(t_1+t_2) = \Phi(t_1) * \Phi(t_2) = \Phi(t_2) * \Phi(t_1)$, $\Phi(t)^n = \Phi(nt)$,

$\Phi(t_2-t_1)\Phi(t_1-t_0) = \Phi(t_2-t_0) = \Phi(t_1-t_0)\Phi(t_2-t_1)$

$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$, $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$, $e^{At}e^{-At} = I$, $e^{(A+B)t} = e^{At}e^{Bt}$ if $AB = BA$

* Evaluating $\frac{o/p}{i/p}$ relation using S.S. Representation:

Since $\dot{\underline{x}} = A\underline{x} + Bu$ and $y = C\underline{x} + Du$

$$sI\underline{x} - A\underline{x} = Bu \quad \longrightarrow \quad y = C(sI-A)^{-1}Bu + Du$$

$$(sI-A)\underline{x} = Bu \qquad y = \left(C(sI-A)^{-1}B + D\right)u$$

$$\therefore \underline{x} = (sI-A)^{-1}Bu \qquad \boxed{\therefore \frac{o/p}{i/p} = \frac{y(s)}{u(s)} = C(sI-A)^{-1}B + D}$$

Note: Signal Flow Graph is one of the ways to extract S.S. equations.

Ex:- Given $y'' + 3y' + 2y = x$ it is required to find its canonical s.s. rep., check for $\frac{o/p(s)}{i/p(s)}$, & calculate e^{At} to find homogenous solution.

solution:- $s^2y + 3sy + 2y = x \Rightarrow \frac{y(s)}{x(s)} = \frac{1}{s^2+3s+2}$, $n = 2$ order

$\therefore A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$

then $\dot{\underline{x}} = \begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$, $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D$

$\therefore \dot x_1 = x_2$, $\dot x_2 = -2x_1 - 3x_2 + u$, $y = x_1$

to check: $\frac{y(s)}{u(s)} = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$

$$\frac{y(s)}{u(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} * \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s^2+3s+2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{y(s)}{u(s)} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ s \end{pmatrix}}{s^2+3s+2} = \frac{1}{s^2+3s+2}$$

to find $e^{At} = \mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{(s+1)} - \frac{1}{(s+2)} \\ \frac{-2}{(s+1)} + \frac{2}{(s+2)} & \frac{-1}{(s+1)} + \frac{2}{(s+2)} \end{bmatrix}$

$\therefore e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$ then $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{At}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ homogenous state Eq. Solution

Then $y(t) = x_1(t)$ System o/p solution.

Question: if non-homogenous solution is required for u = unit step = 1, what will the solution be?

* Methods for finding $\Phi(t) = e^{At} = \mathcal{L}^{-1}(sI-A)^{-1}$

1- Diagonalization: if A is diagonal $\left[ex: \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\right]$ then $e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$ which is easy even for higher order, but transformation of A, B, C, D into diagonal wise will not be an easy job. [A "diagonal" $= P^{-1}AP$, $B = P^{-1}B$, $C^T = C^TP$] i.e, one should find P-matrix to diagonalize. " غير عفوية "

2. If the D.E of the System is available, Canonical S.S. Representation form is easy to extract, then one should find e^{At}. " عفوية "

3. If one analyse a certain system using state-Representation a Random form of S.S. Representation will be obtained, then e^{At} should be calculated. "عفوية"

Any way for 3-point above A, B, C, D matricies will be set, and e^{At} should be calculated either by $\mathcal{L}^{-1}(sI-A)^{-1}$, Cayley-Hamilton theorem, Sylvester's criterion, and Fadeev algorithm.

* Cayley-Hamilton Theorem for finding $\Phi(t) = e^{At}$:-

Here, mainly to compute a polynomial $f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots$ where, $\alpha_0, \alpha_1, \alpha_2 \cdots$ are found from function $g_n(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 \cdots$ with λ's are the eigen values of matrix A.

Ex:- for $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, find A^{100} using Cayley-Hamilton theorem.

$f(A)$

$f(\lambda_1) = g(\lambda_1)$ $\qquad$ $g(\lambda_1) = \alpha_0 + \alpha_1\lambda_1$ $\quad$ "less by 1 from A order"

$f(\lambda_2) = g(\lambda_2)$ $\qquad$ $g(\lambda_2) = \alpha_0 + \alpha_1\lambda_2$

Solve for α_0, α_1 then $f(A) = \alpha_0 I + \alpha_1 A$.

$|\lambda I - A| = 0 \Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \lambda & -1 \\ -1 & \lambda-2 \end{bmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0$

$\lambda_1 = 1+\sqrt{2}$, $\lambda_2 = 1-\sqrt{2}$, $f(\lambda_1) = g(\lambda_1) = \alpha_0 + \alpha_1\lambda_1 \Rightarrow (1+\sqrt{2})^{100} = \alpha_0 + \alpha_1(1+\sqrt{2}) \cdots (1)$

$f(\lambda_2) = g(\lambda_2) = \alpha_0 + \alpha_1\lambda_2 \Rightarrow (1-\sqrt{2})^{100} = \alpha_0 + \alpha_1(1-\sqrt{2}) \cdots (2)$

Solve 1&2 for α_0, α_1 then $A^{100} = f(A) = \alpha_0 I + \alpha_1 A$.